Blind deconvolution by optimizing over a quotient manifold

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January 29, 2017

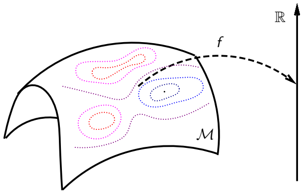
This is joint work with Paul Hand at Rice university

Riemannian Optimization

Problem: Given $f(x) : \mathcal{M} \rightarrow \mathbb{R}$,
solve

$$\min_{x \in \mathcal{M}} f(x)$$

where $\mathcal{M}$ is a Riemannian manifold.



What is a Riemannian manifold? manifold + Riemannian metric

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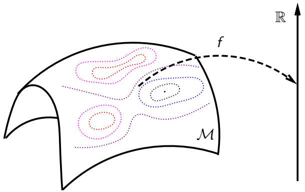

What is a Riemannian manifold? manifold + Riemannian metric

Examples of manifolds:

- $\{X \in \mathbb{R}^{n \times p} \mid X^T X = I_p\}$
- $\{X \in \mathbb{R}^{m \times n} \mid \mathrm{rank}(X) = p\}$
- $\{X \in \mathbb{R}^{n \times n} \mid X = X^T, X \succ 0\}$
- All p-dimensional linear subspaces of $\mathbb{R}^n$

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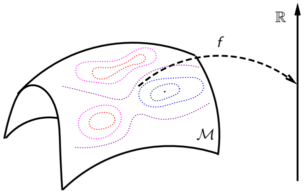

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Riemannian metric:

- Inner product on tangent spaces
- Define angles and lengths

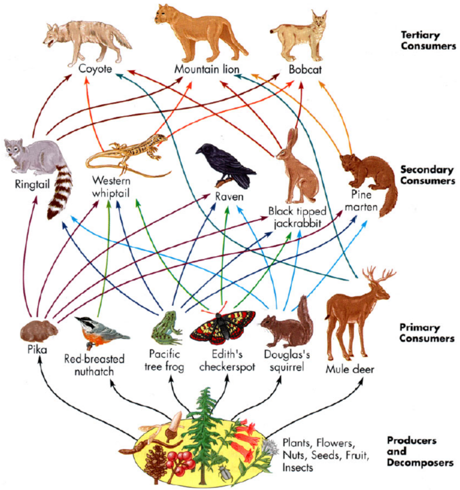
Applications

- Role model extraction [MHB+16]
- Geometric mean of symmetric positive definite matrices [YHAG17]
- Elastic shape analysis of curves [SKJJ11, HGSA15]
- Blind deconvolution [HH17]

Application: Role model extraction

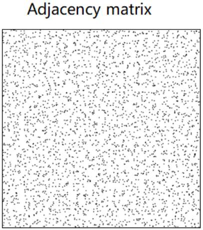
Network Topology

- Role Structures
- Interaction with nodes in either the same role or different roles.

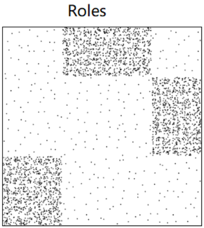


Application: Role model extraction

$A =$



$\overset{P?}{\Longrightarrow} PAP^T =$



- Direct methods (Optimize a cost function over A directly)
- Indirect methods
  - Similarity matrix of the neighborhood pattern similarity measure
  - Cluster highly similar nodes together to extract role partition

Application: Role model extraction

Indirect methods

- Similarity matrix of the neighborhood pattern similarity Measure;
- Cluster highly similar nodes together to extract role partition;

- Find a low-rank approximation of the similarity matrix (Optimization on fixed-rank manifold);
- Use the factor in the low-rank matrix to find roles;

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Indirect methods

- Similarity matrix of the neighborhood pattern similarity Measure;
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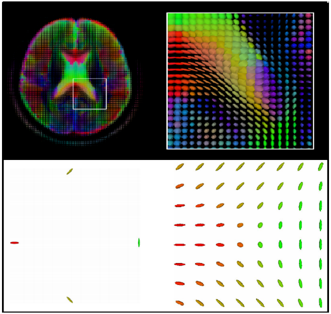
Application: Symmetric Positive Definite (SPD) Matrices

Symmetric positive definite (SPD) matrices are fundamental objects in various domains.

- Object recognition
- Human detection and tracking
- Diffusion tensor magnetic resonance imaging



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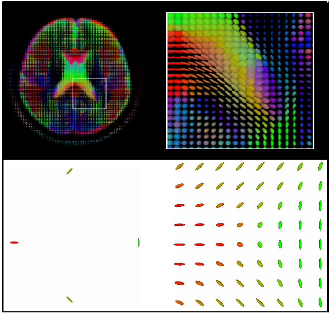
Averaging SPD matrices (Geometric mean)

- Aggregate noisy measurements
- Subtask in interpolation methods



Application: Symmetric Positive Definite (SPD) Matrices

The desired properties of a geometric mean are given in the ALM[1] list, some of which are

- if $A_1, \ldots, A_k$ commute, then $G(A_1, \ldots, A_k) = (A_1 \ldots A_k)^{\frac{1}{k}}$;
- $G(A_{\pi(1)}, \ldots, A_{\pi(k)}) = G(A_1, \ldots, A_k)$, with π a permutation of $(1, \ldots, k)$;
- $G(A_1, \ldots, A_k) = G\left(A_1^{-1}, \ldots A_k^{-1}\right)^{-1}$;
- $\det G(A_1, \ldots, A_k) = (\det A_1 \ldots \det A_k)^{\frac{1}{k}}$;

where $A_1, \ldots, A_k$ are SPD matrices, and $G(\cdot, \ldots, \cdot)$ denotes the geometric mean of arguments.

[1] T. Ando, C.-K. Li, and R. Mathias, Geometric means, *Linear Algebra and Its Applications*, 385:305-334, 2004

Application: Geometric Mean of Symmetric Positive Definite Matrices

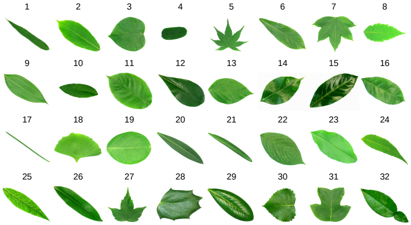
One geometric mean is the Karcher mean of the manifold of SPD matrices with the affine invariant metric, i.e.,

$$G(A_1, \dots, A_k) = \arg\min_{X \in \mathbb{S}_+^n} \frac{1}{2k} \sum_{i=1}^{k} \mathrm{dist}^2(X, A_i),$$

where $\mathrm{dist}(X, Y) = \| \log(X^{-1/2} Y X^{-1/2}) \|_F$ is the distance under the Riemannian metric

$$g(\eta_X, \xi_X) = \mathrm{trace}(\eta_X X^{-1} \xi_X X^{-1}).$$

Application: Elastic Shape Analysis of Curves



- Classification [LKS+12, HGSA15]
- Face recognition [DBS+13]

Application: Elastic Shape Analysis of Curves

- Elastic shape analysis invariants:
  - Rescaling
  - Translation
  - Rotation
  - Reparametrization
- The shape space is a quotient space

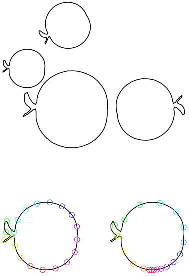
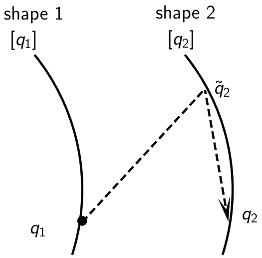


Figure: All are the same shape.

Application: Elastic Shape Analysis of Curves



- Optimization problem $\min_{q_2 \in [q_2]} \mathrm{dist}(q_1, q_2)$ is defined on a Riemannian manifold
- Computation of a geodesic between two shapes
- Computation of Karcher mean of a population of shapes

ROPTLIB

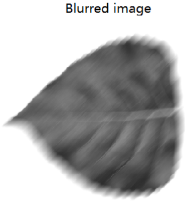
Riemannian manifold optimization library (ROPTLIB) is used to optimize a function on a manifold.

- Most state-of-the-art methods;
- Commonly-encountered manifolds;
- Written in C++;
- Interfaces with Matlab, Julia and R;
- BLAS and LAPACK;
- `www.math.fsu.edu/~whuang2/Indices/index_ROPTLIB.html`

Blind deconvolution

[Blind deconvolution]

Blind deconvolution is to recover two unknown signals from their convolution.



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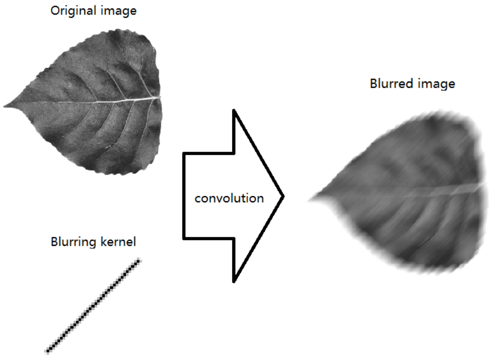

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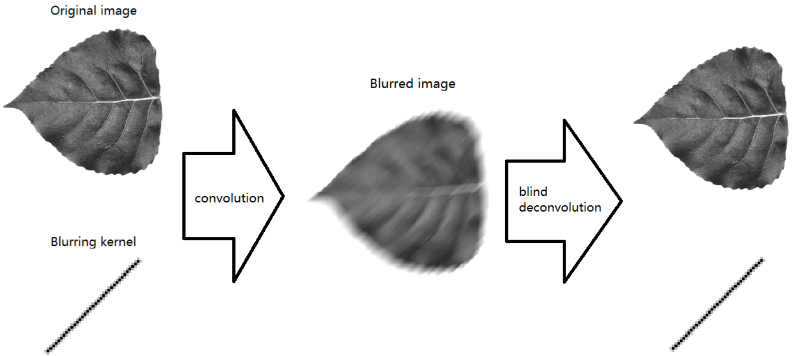

Problem Statement

[Blind deconvolution (Discretized version)]

Blind deconvolution is to recover two unknown signals $\mathbf{w} \in \mathbb{C}^L$ and $\mathbf{x} \in \mathbb{C}^L$ from their convolution $\mathbf{y} = \mathbf{w} * \mathbf{x} \in \mathbb{C}^L$.

- We only consider circular convolution:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \vdots \\ \mathbf{y}_L \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_L & \mathbf{w}_{L-1} & \dots & \mathbf{w}_2 \\ \mathbf{w}_2 & \mathbf{w}_1 & \mathbf{w}_L & \dots & \mathbf{w}_3 \\ \mathbf{w}_3 & \mathbf{w}_2 & \mathbf{w}_1 & \dots & \mathbf{w}_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_L & \mathbf{w}_{L-1} & \mathbf{w}_{L-2} & \dots & \mathbf{w}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_L \end{bmatrix}$$

- Let $y = \mathbf{F}\mathbf{y}$, $w = \mathbf{F}\mathbf{w}$, and $x = \mathbf{F}\mathbf{x}$, where $\mathbf{F}$ is the DFT matrix;
- $y = w \odot x$, where $\odot$ is the Hadamard product, i.e., $y_i = w_i x_i$.

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- $y = w \odot x$, where $\odot$ is the Hadamard product, i.e., $y_i = w_i x_i$.
- Equivalent question: Given y, find w and x.

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- Assumption: w and x are in known subspaces, i.e., $w = Bh$ and $x = \overline{Cm}$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$;

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Problem under the assumption

Given $y \in \mathbb{C}^L$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$, find $h \in \mathbb{C}^K$ and $m \in \mathbb{C}^N$ so that

$$y = Bh \odot \overline{Cm} = \mathrm{diag}(Bhm^* C^*).$$

Related work

Find h, m, s. t. $y = \mathrm{diag}(Bhm^* C^*)$;

- Ahmed et al. [ARR14][2]
  - Convex problem:

$$\min_{X \in \mathbb{C}^{K \times N}} \|X\|_n, \text{ s. t. } y = \mathrm{diag}(BXC^*),$$

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- Li et al. [LLSW16][3]
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  - Lower successful recovery probability than alternating minimization algorithm empirically.

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Manifold Approach

$$\boxed{\text{Find } h, m, \text{ s. t. } y = \mathrm{diag}(Bhm^* C^*);}$$

- The problem is defined on the set of rank-one matrices (denoted by $\mathbb{C}_1^{K\times N}$), neither $\mathbb{C}^{K\times N}$ nor $\mathbb{C}^K \times \mathbb{C}^N$; Why not work on the manifold directly?

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- Optimization on manifolds: A Riemannian steepest descent method;
  - A representation of $\mathbb{C}_1^{K \times N}$;
  - Representation of directions;
  - A Riemannian metric;
  - Riemannian gradient;

A Representation of $\mathbb{C}_1^{K\times N}$: $\mathbb{C}_*^K \times \mathbb{C}_*^N / \mathbb{C}_*$

- Given $X \in \mathbb{C}_1^{K\times N}$, there exists (h, m) such that $X = hm^*$;
- (h, m) is not unique;
- The equivalent class: $[(h, m)] = \{(ha, ma^{-*}) \mid a \neq 0\}$;
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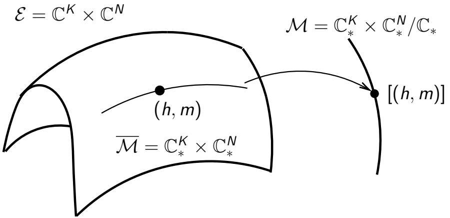


$$\mathbb{C}_*^K \times \mathbb{C}_*^N/\mathbb{C}_* \simeq \mathbb{C}_1^{K\times N}$$

A Representation of $\mathbb{C}_1^{K\times N}$: $\mathbb{C}_*^K \times \mathbb{C}_*^N/\mathbb{C}_*$

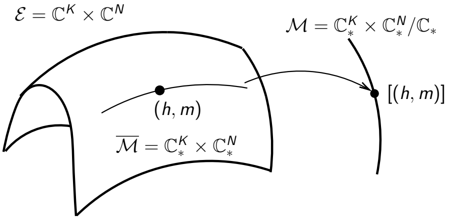
Cost function[5]

- Riemannian approach:

  $$f : \mathbb{C}_*^K \times \mathbb{C}_*^N/\mathbb{C}_* \to \mathbb{R} : [(h, m)] \mapsto \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2.$$

- Approach in [LLSW16]:

  $$\mathfrak{f} : \mathbb{C}^K \times \mathbb{C}^N \to \mathbb{R} : (h, m) \mapsto \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2.$$



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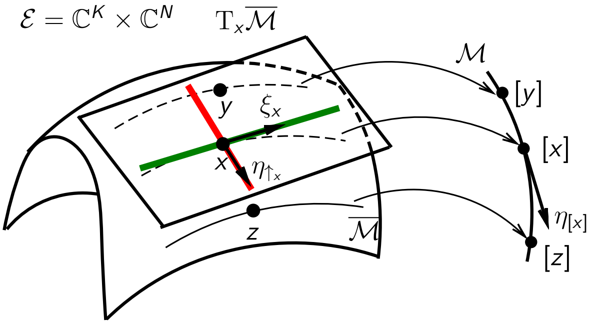
Representation of directions on $\mathbb{C}_*^K \times \mathbb{C}_*^N / \mathbb{C}_*$



- x denotes (h, m);
- Green line: the tangent space of $[x]$;
- Red line (horizontal space at x): orthogonal to the green line;
- Horizontal space at x: a representation of the tangent space of $\mathcal{M}$ at $[x]$;

A Riemannian metric

Riemannian metric:

- Inner product on tangent spaces
- Define angles and lengths

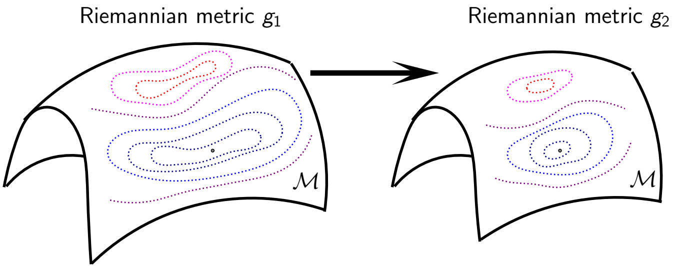


Figure: Changing metric may influence the difficulty of a problem.

A Riemannian metric

$$\min_{[(h,m)]} \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2$$

Idea for choosing a Riemannian metric

The block diagonal terms in the Euclidean Hessian are used to choose the Riemannian metric.

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The block diagonal terms in the Euclidean Hessian are used to choose the Riemannian metric.

- Let $\langle u, v\rangle_2 = \mathrm{Re}(\mathrm{trace}(u^* v))$:

$$\frac{1}{2}\langle \eta_h, \mathrm{Hess}_h f[\xi_h]\rangle_2 = \langle \mathrm{diag}(B\eta_h m^* C^*), \mathrm{diag}(B\xi_h m^* C^*)\rangle_2 \approx \langle \eta_h m^*, \xi_h m^*\rangle_2$$

$$\frac{1}{2}\langle \eta_m, \mathrm{Hess}_m f[\xi_m]\rangle_2 = \langle \mathrm{diag}(Bh\eta_m^* C^*), \mathrm{diag}(Bh\xi_m^* C^*)\rangle_2 \approx \langle h\eta_m^*, h\xi_m^*\rangle_2,$$

where $\approx$ can be derived from some assumptions (given later);

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  where $\approx$ can be derived from some assumptions (given later);
- The Riemannian metric:

$$g\left(\eta_{[x]}, \xi_{[x]}\right) = \langle \eta_h, \xi_h m^* m \rangle_2 + \langle \eta_m^*, \xi_m^* h^* h \rangle_2;$$

Riemannian gradient

- Riemannian gradient
  - A tangent vector: $\mathrm{grad}\, \mathbf{f}([x]) \in \mathrm{T}_{[x]}\, \mathcal{M}$;
  - Satisfies: $\mathrm{D} f([x])[\eta_{[x]}] = g(\mathrm{grad}\, f([x]), \eta_{[x]}), \;\; \forall \eta_{[x]} \in \mathrm{T}_{[x]}\, \mathcal{M}$;
- Represented by a vector in a horizontal space;
- Riemannian gradient:

$$(\mathrm{grad}\, f([(h, m)]))_{\uparrow_{(h,m)}} = Proj\left(\nabla_h f(h, m)(m^* m)^{-1}, \nabla_m f(h, m)(h^* h)^{-1}\right);$$

A Riemannian steepest descent method (RSD)

An implementation of a Riemannian steepest descent method[6]

0. Given (h_0, m_0), step size $\alpha > 0$, and set $k = 0$
1. $d_k = \|h_k\|_2 \|m_k\|_2$, $h_k \leftarrow \sqrt{d_k} \frac{h_k}{\|h_k\|_2}$; $m_k \leftarrow \sqrt{d_k} \frac{m_k}{\|m_k\|_2}$;
2. $(h_{k+1}, m_{k+1}) = (h_k, m_k) - \alpha \left(\frac{\nabla_{h_k} f(h_k, m_k)}{d_k}, \frac{\nabla_{m_k} f(h_k, m_k)}{d_k} \right)$;
3. If not converge, goto Step 2.

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Penalty/Coherence

- Coherence is defined as

$$\mu_h^2 = \frac{L\|Bh\|_\infty^2}{\|h\|_2^2} = \frac{L\max\left(|b_1^* h|^2, |b_2^* h|^2, \dots, |b_L^* h|^2\right)}{\|h\|_2^2};$$

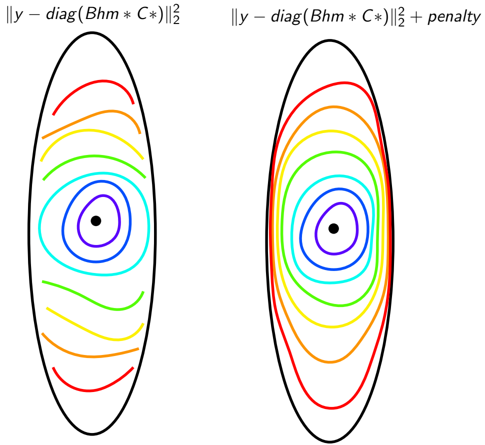
- Coherence at the true solution $[(h_\sharp, m_\sharp)]$
  - influences the probability of recovery
  - Small coherence is preferred

Penalty

Promote low coherence:

$$\rho \sum_{i=1}^{L} G_0 \left(\frac{L |b_i^* h|^2 \|m\|_2^2}{8 d^2 \mu^2} \right),$$

where $G_0(t) = \max(t-1, 0)^2$;



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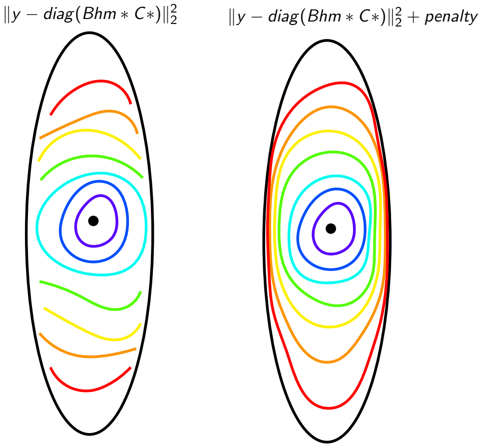
- Ω: ellipsoid;



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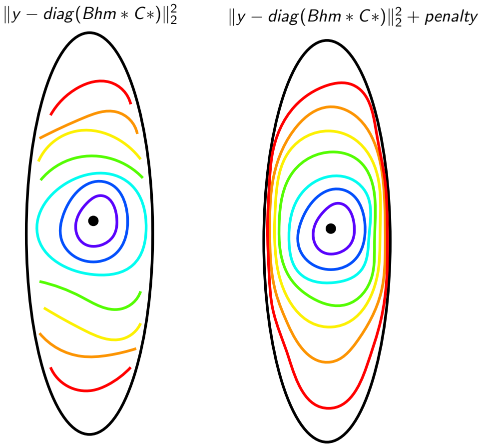
- Unique minimizer in Ω;



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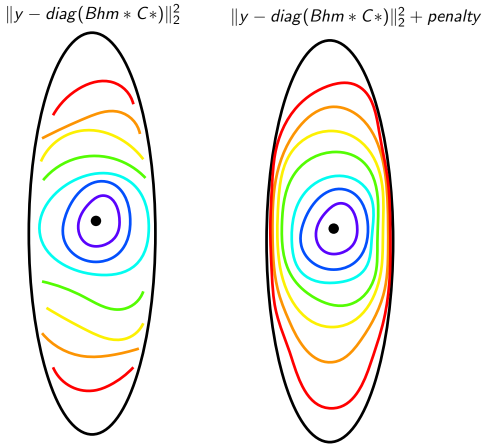
- Ω: ellipsoid;
- Unique minimizer in Ω;
- Initial iterate in Ω;



Penalty

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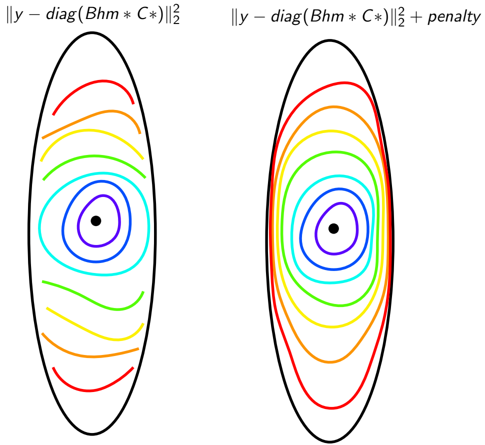
where $G_0(t) = \max(t-1, 0)^2$;

- Ω: ellipsoid;
- Unique minimizer in Ω;
- Initial iterate in Ω;
- Importance of the penalty;



Penalty

- Riemannian approach:

$$\rho \sum_{i=1}^{L} G_0\left(\frac{L|b_i^* h|^2 \|m\|_2^2}{8d^2\mu^2}\right)$$

- [LLSW16]:

$$\rho \left[G_0\left(\frac{\|h\|_2^2}{2d}\right) + G_0\left(\frac{\|m\|_2^2}{2d}\right) + \sum_{i=1}^{L} G_0\left(\frac{L|b_i^* h|^2}{8d\mu^2}\right)\right]$$

- $G_0(t) = \max(t-1, 0)^2$, $[b_1 b_2 \dots b_L]^* = B$;

Penalty

- Riemannian approach:

$$\rho \sum_{i=1}^{L} G_0 \left(\frac{L |b_i^* h|^2 \|m\|_2^2}{8 d^2 \mu^2} \right)$$

- [LLSW16]:

$$\rho \left[G_0 \left(\frac{\|h\|_2^2}{2d} \right) + G_0 \left(\frac{\|m\|_2^2}{2d} \right) + \sum_{i=1}^{L} G_0 \left(\frac{L |b_i^* h|^2}{8 d \mu^2} \right) \right]$$

- $G_0(t) = \max(t-1, 0)^2$, $[b_1 b_2 \dots b_L]^* = B$;

Riemannian approach avoids the two terms.

Initialization

Initialization method [LLSW16]

- $(d, \tilde{h}_0, \tilde{m}_0)$: SVD of $B^* \operatorname{diag}(y) C$;
- Project $(\tilde{h}_0, \tilde{m}_0)$ to a neighborhood of the true solution;
- Initial iterate $[(h_0, m_0)]$;

Numerical Results

- Synthetic tests
  - Efficiency
  - Probability of successful recovery
- Image deblurring

Efficiency

Table: Comparisons of efficiency

	$L = 400, K = N = 50$			$L = 600, K = N = 50$		
Algorithms	[LLSW16]	[LWB13]	R-SD	[LLSW16]	[LWB13]	R-SD
nBh/nCm	351	718	208	162	294	122
$nFFT$	870	1436	518	401	588	303
$RMSE$	2.22_{-8}	3.67_{-8}	2.20_{-8}	1.48_{-8}	2.34_{-8}	1.42_{-8}

- An average of 100 random runs
- nBh/nCm: the numbers of Bh and Cm multiplication operations respectively
- nFFT: the number of Fourier transform
- RMSE: the relative error $\frac{\|hm^* - h_\sharp m_\sharp^*\|_F}{\|h_\sharp\|_2 \|m_\sharp\|_2}$

[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016
[LWB13]: K. Lee et. al., Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization *preprint arXiv:1312.0525*, 2013

Probability of successful recovery

- Success if $\frac{\|hm^* - h_\sharp m_\sharp^*\|_F}{\|h_\sharp\|_2 \|m_\sharp\|_2} \leq 10^{-2}$

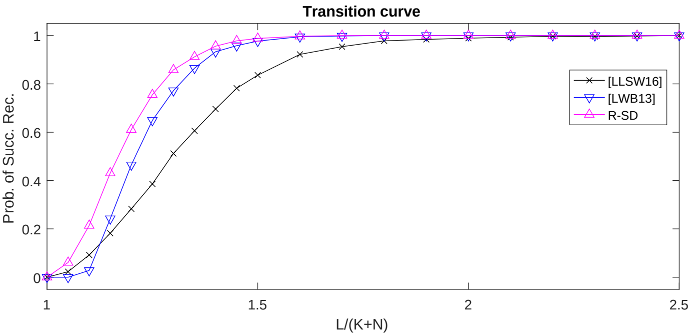


Figure: Empirical phase transition curves for 1000 random runs.

[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016
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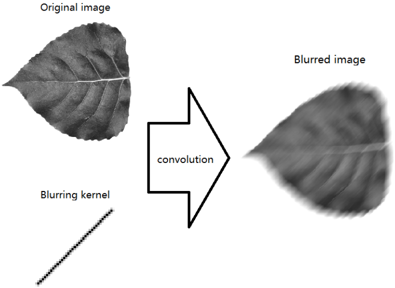
Image deblurring



- Original image [WBX$^+$07]: 1024-by-1024 pixels
- Motion blurring kernel (Matlab: fspecial('motion', 50, 45))

Image deblurring

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis
- Support of the blurring kernel is learned from the blurred image

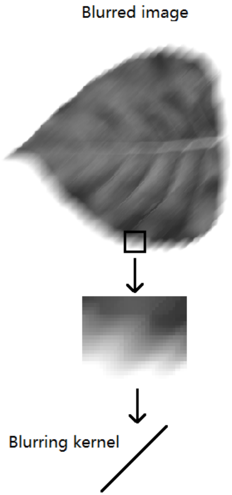


Image deblurring

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis

  Use the blurred image to learn the dominated basis: **C**.

- Support of the blurring kernel is learned from the blurred image

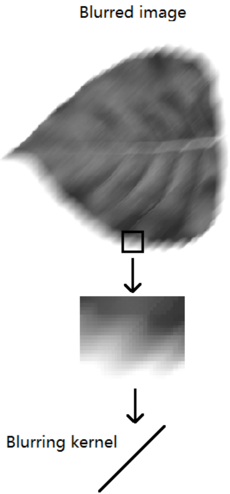


Image deblurring

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis

  Use the blurred image to learn the dominated basis: **C**.

- Support of the blurring kernel is learned from the blurred image

  Suppose the support of the blurring kernel is known: **B**.

- $L = 1048576, K = 109,$
  $N = 5000, 20000, 80000$

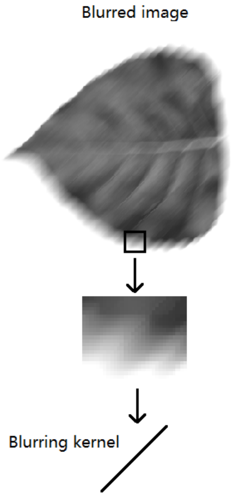


Image deblurring

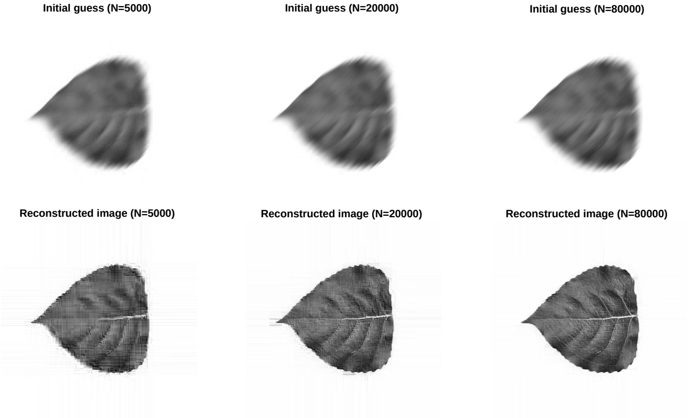


Figure: Initial guess by running power method for 50 iterations and the reconstructed image for $N = 5000, 20000$, and 80000.

Image deblurring

Table: Computational costs for multiple values of N on the image deblurring

N	nBh/nCm	$nFFT$	$relres$	$relerr$	t
5000	535	1330	4.8_{-3}	5.7_{-2}	170
20000	546	1358	2.1_{-3}	5.3_{-2}	173
80000	452	1124	8.0_{-4}	5.0_{-2}	144

- *relres*: $\|y - \mathrm{diag}(Bhm^* C^*)\|_2 / \|y\|_2$;
- *relerr*: $\left\| \mathbf{y}_o - \frac{\|\mathbf{y}\|}{\|\mathbf{y}_f\|} \mathbf{y}_f \right\| / \|\mathbf{y}_o\|$
- $\mathbf{y}_f$: the vector by reshaping the reconstructed image
- $\mathbf{y}_o$: the vector by reshaping the original image

Theoretical Results

Mathematical model:

- The entries in C are drawn from Gaussian distribution; and
- B satisfies $B^* B = I_K$ and $\|b_i\|_2^2 \leq \phi \frac{K}{L}, i = 1, \dots, L$ for some constant ϕ.

Theoretical Results

Initialization [LLSW16][7]

If $L \geq C_\gamma(\mu^2 + \sigma^2)\max(K, N)\log^2(L)/\varepsilon$, then with high probability, it holds that

$$[(h_0, m_0)] \in \Omega_{\frac{1}{2}\mu} \cap \Omega_{\frac{2}{5}\varepsilon},$$

where $\Omega_\mu = \{[(h, m)] \mid \sqrt{L}\|Bh\|_\infty \|m\|_2 \leq 4 d_* \mu\}$,
$\Omega_\varepsilon = \{[(h, m)] \mid \|hm^* - h_\sharp m_\sharp^*\|_F \leq \varepsilon d_*\}$, and $(h_\sharp, m_\sharp)$ is the true solution.

Large enough number of measurements $\Longrightarrow$ the initial point in a small neighborhood of the true solution.

[7] X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

Theoretical Results

Convergence analysis

Suppose $L \geq C_\gamma(\mu^2 + \sigma^2) \max(K, N) \log^2(L)/\varepsilon^2$ and the initialization $[(h_0, m_0)] \in \Omega_{\frac{1}{2}\mu} \cap \Omega_{\frac{2}{5}\varepsilon}$. Then with high probability, it holds that

$$\|h_k m_k^* - h_\sharp m_\sharp^*\|_F \leq \frac{2}{3}\left(1 - \frac{\alpha}{3000}\right)^{k/2} \varepsilon d_*,$$

where α is a small enough fixed step size.

i) Large enough number of measurements; ii) the initial point in a small neighborhood of the true solution $\Longrightarrow$ the Riemannian method converges linearly to the true solution.

Theoretical Results

Riemannian Hessian

Suppose $L \geq C_\gamma \max(K, \mu_h^2 N) \log^2(L)$. Then with high probability, it holds that

$$\frac{9d_*^2}{5} \leq \lambda_i \leq \frac{22d_*^2}{5}$$

for all i, where λ_i are eigenvalues of the Riemannian Hessian $\mathrm{Hess}\, f$ at the true solution.

The Riemannian Hessian $f \circ R$ is well-conditioned near the true solution.

Conclusion

Blind deconvolution by optimizing over a quotient manifold

- A Riemannian steepest descent method
- Simple implementation
- Recovery guarantee
- Superior numerical performance

Thank you

Thank you!

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