

Riemannian Optimization with its Application to Blind Deconvolution Problem

Wen Huang

Rice University

April 27, 2018

Experience

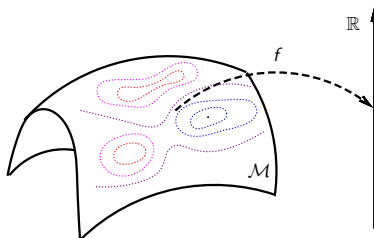
- Ph.D of Applied and Computational Mathematics from Florida State University (2014)
- Postdoctoral fellow in the department of Scientific Computing at Florida State University (2014.5-2014.8)
- Postdoctoral fellow in the department of Mathematical Engineering at Université catholique de Louvain (2014.9-2016.6)
- Pfeiffer postdoctoral instructor in the department of Computational and Applied Mathematics at Rice University (2016.7-present)

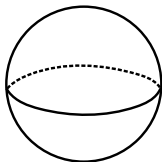
Riemannian Optimization

Problem: Given $f(x) : \mathcal{M} \rightarrow \mathbb{R}$, solve

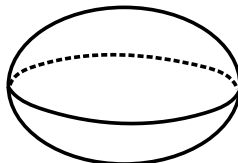
$$\min_{x \in \mathcal{M}} f(x)$$

where \mathcal{M} is a Riemannian manifold.





Sphere

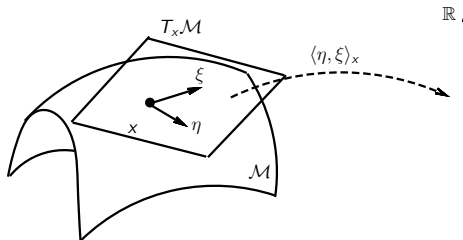


Ellipsoid

- Stiefel manifold: $\text{St}(p, n) = \{X \in \mathbb{R}^{n \times p} | X^T X = I_p\}$
- Grassmann manifold: Set of all p -dimensional subspaces of \mathbb{R}^n
- Set of fixed rank m -by- n matrices
- And many more

Riemannian Manifolds

Roughly, a Riemannian manifold \mathcal{M} is a smooth set with a smoothly-varying inner product on the tangent spaces.

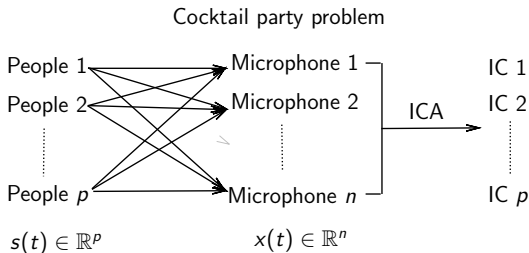


Applications

Three applications are used to demonstrate the importance of the Riemannian optimization:

- Independent component analysis [CS93]
- Matrix completion problem [Van13, HAGH16]
- Elastic shape analysis of curves [SKJJ11, HGSA15]

Application: Independent Component Analysis



- Observed signal is $x(t) = As(t)$
- One approach:
 - Assumption: $E\{s(t)s(t+\tau)\}$ is diagonal for all τ
 - $C_\tau(x) := E\{x(t)x(t+\tau)^T\} = AE\{s(t)s(t+\tau)^T\}A^T$

Application: Independent Component Analysis

- Minimize joint diagonalization cost function on the Stiefel manifold [TI06]:

$$f : \text{St}(p, n) \rightarrow \mathbb{R} : V \mapsto \sum_{i=1}^N \|V^T C_i V - \text{diag}(V^T C_i V)\|_F^2.$$

- C_1, \dots, C_N are covariance matrices and $\text{St}(p, n) = \{X \in \mathbb{R}^{n \times p} | X^T X = I_p\}$.

Application: Matrix Completion Problem

Matrix completion problem

| | Movie 1 | Movie 2 | | Movie n |
|----------|---------|---------|---|-----------|
| User 1 | | 1 | | 4 |
| User 2 | 3 | 5 | | 4 |
| | | 5 | 1 | |
| User m | | 2 | | 5 |
| | | | | 3 |

Rate matrix M

- The matrix M is sparse
- The goal: complete the matrix M

Application: Matrix Completion Problem

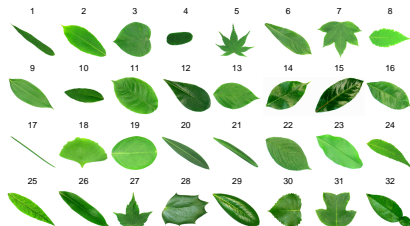
$$\begin{array}{c} \text{movies} \end{array} \begin{pmatrix} a_{11} & & & a_{14} \\ & & & a_{24} \\ & & a_{33} & \\ a_{41} & & & \\ & a_{52} & a_{53} & \end{pmatrix} = \begin{array}{c} \text{meta-user} \end{array} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{pmatrix} \begin{array}{c} \text{meta-movie} \end{array} \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix}$$

- Minimize the cost function

$$f : \mathbb{R}_r^{m \times n} \rightarrow \mathbb{R} : X \mapsto f(X) = \|P_\Omega M - P_\Omega X\|_F^2.$$

- $\mathbb{R}_r^{m \times n}$ is the set of m -by- n matrices with rank r . It is known to be a Riemannian manifold.

Application: Elastic Shape Analysis of Curves



- Classification
[LKS⁺12, HGSA15]
- Face recognition
[DBS⁺13]



Application: Elastic Shape Analysis of Curves

- Elastic shape analysis invariants:
 - Rescaling
 - Translation
 - Rotation
 - Reparametrization
- The shape space is a quotient space

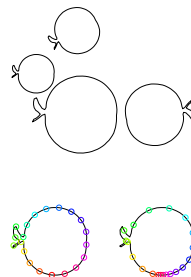
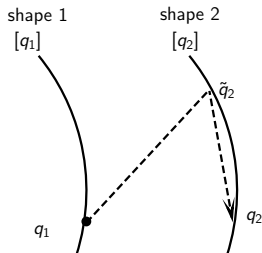


Figure: All are the same shape.



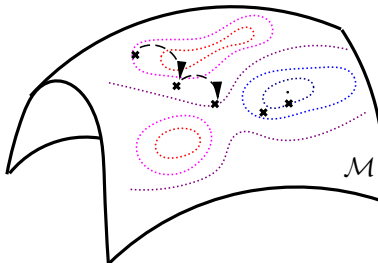
- Optimization problem $\min_{q_2 \in [q_2]} \text{dist}(q_1, q_2)$ is defined on a Riemannian manifold
- Computation of a geodesic between two shapes
- Computation of Karcher mean of a population of shapes

More Applications

- Role model extraction [MHB⁺16]
- Computations on SPD matrices [YHAG17]
- Phase retrieval problem [HGZ17]
- Blind deconvolution [HH17]
- Synchronization of rotations [Hua13]
- Computations on low-rank tensor
- Low-rank approximate solution for Lyapunov equation

Comparison with Constrained Optimization

- All iterates on the manifold
- Convergence properties of unconstrained optimization algorithms
- No need to consider Lagrange multipliers or penalty functions
- Exploit the structure of the constrained set



Iterations on the Manifold

Consider the following generic update for an iterative Euclidean optimization algorithm:

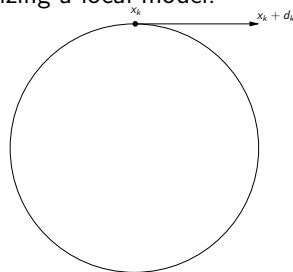
$$x_{k+1} = x_k + \Delta x_k = x_k + \alpha_k s_k .$$

This iteration is implemented in numerous ways, e.g.:

- Steepest descent: $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$
- Newton's method: $x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)$
- Trust region method: Δx_k is set by optimizing a local model.

Riemannian Manifolds Provide

- Riemannian concepts describing **directions** and **movement** on the manifold
- Riemannian analogues for **gradient** and **Hessian**



Riemannian gradient and Riemannian Hessian

Definition

The **Riemannian gradient** of f at x is the unique tangent vector in $T_x M$ satisfying $\forall \eta \in T_x M$, the directional derivative

$$Df(x)[\eta] = \langle \text{grad } f(x), \eta \rangle$$

and $\text{grad } f(x)$ is the direction of steepest ascent.

Definition

The **Riemannian Hessian** of f at x is a symmetric linear operator from $T_x M$ to $T_x M$ defined as

$$\text{Hess } f(x) : T_x M \rightarrow T_x M : \eta \rightarrow \nabla_\eta \text{grad } f,$$

where ∇ is the affine connection.

Retractions

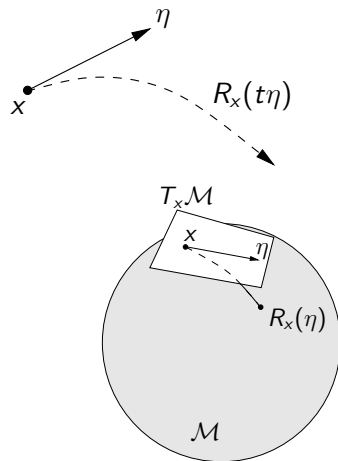
| Euclidean | Riemannian |
|--------------------------------|--------------------------------------|
| $x_{k+1} = x_k + \alpha_k d_k$ | $x_{k+1} = R_{x_k}(\alpha_k \eta_k)$ |

Definition

A **retraction** is a mapping R from TM to M satisfying the following:

- R is continuously differentiable
- $R_x(0) = x$
- $D R_x(0)[\eta] = \eta$

- maps tangent vectors back to the manifold
- defines curves in a direction



Categories of Riemannian optimization methods

Retraction-based: local information only

Line search-based: use local tangent vector and $R_x(t\eta)$ to define line

- Steepest decent
- Newton

Local model-based: series of flat space problems

- Riemannian trust region Newton (RTR)
- Riemannian adaptive cubic overestimation (RACO)

Categories of Riemannian optimization methods

Retraction and transport-based: information from multiple tangent spaces

- Nonlinear conjugate gradient: multiple tangent vectors
- Quasi-Newton e.g. Riemannian BFGS: transport operators between tangent spaces

Additional element required for optimizing a cost function (M, g) :

- formulas for combining information from multiple tangent spaces.

Vector Transports

Vector Transport

- Vector transport: Transport a tangent vector from one tangent space to another
- $\mathcal{T}_{\eta_x} \xi_x$, denotes transport of ξ_x to tangent space of $R_x(\eta_x)$. R is a retraction associated with \mathcal{T}

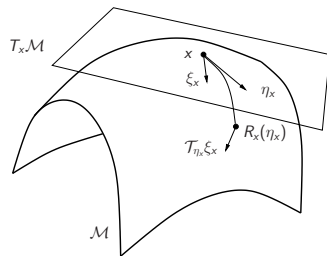


Figure: Vector transport.

Retraction/Transport-based Riemannian Optimization

Given a retraction and a vector transport, we can generalize many Euclidean methods to the Riemannian setting. Do the Riemannian versions of the methods work well?

Retraction/Transport-based Riemannian Optimization

Given a retraction and a vector transport, we can generalize many Euclidean methods to the Riemannian setting. Do the Riemannian versions of the methods work well?

No

- Lose many theoretical results and important properties;
- Impose restrictions on retraction/vector transport;

Retraction/Transport-based Riemannian Optimization

Benefits

- Increased generality does not compromise the **important theory**
- Less expensive than or similar to previous approaches
- May provide theory to explain behavior of algorithms specifically developed for a particular application – or closely related ones

Possible Problems

- May be inefficient compared to algorithms that exploit application details

Some History of Optimization On Manifolds (I)

[Luenberger \(1973\)](#), *Introduction to linear and nonlinear programming*. Luenberger mentions the idea of performing line search along geodesics, “which we would use if it were computationally feasible (which it definitely is not)”. Rosen (1961) essentially anticipated this but was not explicit in his Gradient Projection Algorithm.

[Gabay \(1982\)](#), *Minimizing a differentiable function over a differential manifold*. Steepest descent along geodesics; Newton’s method along geodesics; Quasi-Newton methods along geodesics. On Riemannian submanifolds of \mathbb{R}^n .

[Smith \(1993-94\)](#), *Optimization techniques on Riemannian manifolds*. Levi-Civita connection ∇ ; Riemannian exponential mapping; parallel translation.

Some History of Optimization On Manifolds (II)

The “pragmatic era” begins:

[Manton \(2002\)](#), *Optimization algorithms exploiting unitary constraints*

“The present paper breaks with tradition by not moving along geodesics”. The geodesic update $\text{Exp}_x \eta$ is replaced by a projective update $\pi(x + \eta)$, the *projection* of the point $x + \eta$ onto the manifold.

[Adler, Dedieu, Shub, et al. \(2002\)](#), *Newton’s method on Riemannian manifolds and a geometric model for the human spine*. The exponential update is relaxed to the general notion of *retraction*. The geodesic can be replaced by any (smoothly prescribed) curve tangent to the search direction.

[Absil, Mahony, Sepulchre \(2007\)](#) Nonlinear conjugate gradient using retractions.

Some History of Optimization On Manifolds (III)

Theory, efficiency, and library design improve dramatically:

[Absil, Baker, Gallivan \(2004-07\)](#), Theory and implementations of Riemannian Trust Region method. Retraction-based approach. Matrix manifold problems, software repository

<http://www.math.fsu.edu/~cbaker/GenRTR>

Anasazi Eigenproblem package in Trilinos Library at Sandia National Laboratory

[Absil, Gallivan, Qi \(2007-10\)](#), Basic theory and implementations of Riemannian BFGS and Riemannian Adaptive Cubic Overestimation. Parallel translation and Exponential map theory, Retraction and vector transport empirical evidence.

Some History of Optimization On Manifolds (IV)

Ring and With (2012), combination of differentiated retraction and isometric vector transport for convergence analysis of RBFGS

Absil, Gallivan, Huang (2009-2017), Complete theory of Riemannian Quasi-Newton and related transport/retraction conditions, Riemannian SR1 with trust-region, RBFGS on partly smooth problems, A C++ library: <http://www.math.fsu.edu/~whuang2/ROPTLIB>

Sato, Iwai (2013-2015), Zhu (2017), Global convergence analysis for Riemannian conjugate gradient methods

Bonnabel (2011), Sato, Kasai, Mishra(2017) Riemannian stochastic gradient descent method.

Many people Application interests increase noticeably

Methods by Our Group

- Riemannian Steepest Descent [AMS08]
- Riemannian conjugate gradient [AMS08]
- Riemannian Trust Region Newton [ABG07]: global, quadratic convergence
- Riemannian Broyden Family [HGA15, HAG18] : global (convex), superlinear convergence
- Riemannian Trust Region SR1 [HAG15]: global, $(d + 1)$ –superlinear convergence
- For large problems
 - Limited memory RTRSR1
 - Limited memory RBFGS

Library by Our Group

Riemannian manifold optimization library (ROPTLIB) is used to optimize a function on a manifold.

- Most state-of-the-art methods;
- Commonly-encountered manifolds;
- Written in C++;
- Interfaces with Matlab, Julia and R;
- BLAS and LAPACK;
- www.math.fsu.edu/~whuang2/Indices/index_ROPTLIB.html

Current/Future Work on Riemannian methods

- Manifold and inequality constraints
- Discretization of infinite dimensional manifolds and the convergence/accuracy of the approximate minimizers – specific to a problem and extracting general conclusions
- Partly smooth cost functions on Riemannian manifold
- Limited-memory quasi-Newton methods on manifolds

Blind deconvolution

[Blind deconvolution]

Blind deconvolution is to recover two unknown signals from their convolution.

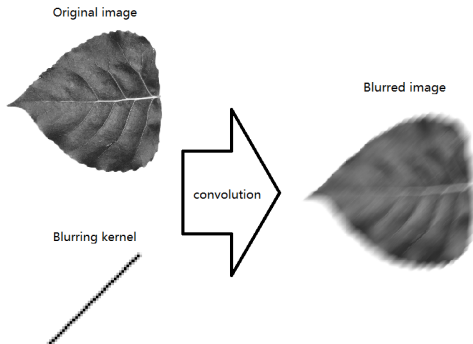
Blurred image



Blind deconvolution

[Blind deconvolution]

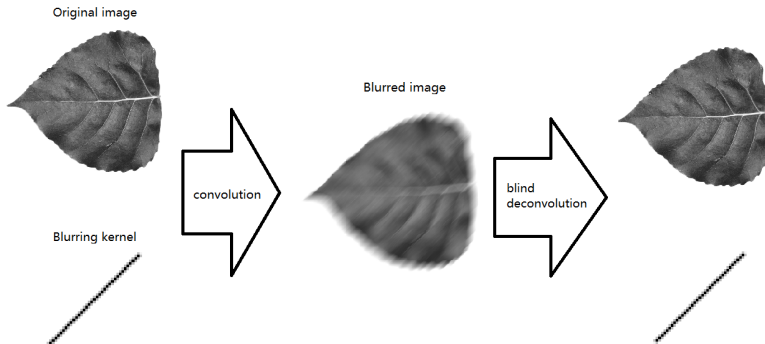
Blind deconvolution is to recover two unknown signals from their convolution.



Blind deconvolution

[Blind deconvolution]

Blind deconvolution is to recover two unknown signals from their convolution.



Problem Statement

[Blind deconvolution (Discretized version)]

Blind deconvolution is to recover two unknown signals $\mathbf{w} \in \mathbb{C}^L$ and $\mathbf{x} \in \mathbb{C}^L$ from their convolution $\mathbf{y} = \mathbf{w} * \mathbf{x} \in \mathbb{C}^L$.

- We only consider circular convolution;
- Equivalent problem: Given y , find w and x such that $y = w \otimes x$;
- Problem under the assumption: Given $y \in \mathbb{C}^L$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$, find $h \in \mathbb{C}^K$ and $m \in \mathbb{C}^N$ such that

$$y = Bh \odot \overline{Cm} = \text{diag}(Bhm^* C^*).$$

Related work

Find h, m , s. t. $y = \text{diag}(Bhm^*C^*)$;

- Ahmed et al. [ARR14]¹

- Convex problem:

$$\min_{X \in \mathbb{C}^{K \times N}} \|X\|_n, \text{ s. t. } y = \text{diag}(BXC^*),$$

where $\|\cdot\|_n$ denotes the nuclear norm, and $X = hm^*$;

¹A. Ahmed, B. Recht, and J. Romberg, Blind deconvolution using convex programming, *IEEE Transactions on Information Theory*, 60:1711-1732, 2014

Related work

Find h, m , s. t. $y = \text{diag}(Bhm^*C^*)$;

- Ahmed et al. [ARR14]¹

- Convex problem:

$$\min_{X \in \mathbb{C}^{K \times N}} \|X\|_n, \text{ s. t. } y = \text{diag}(BXC^*),$$

where $\|\cdot\|_n$ denotes the nuclear norm, and $X = hm^*$;

- (Theoretical result): the unique minimizer high probability the true solution;

¹A. Ahmed, B. Recht, and J. Romberg, Blind deconvolution using convex programming, *IEEE Transactions on Information Theory*, 60:1711-1732, 2014

Related work

Find h, m , s. t. $y = \text{diag}(Bhm^*C^*)$;

■ Ahmed et al. [ARR14]¹

- Convex problem:

$$\min_{X \in \mathbb{C}^{K \times N}} \|X\|_n, \text{ s. t. } y = \text{diag}(BXC^*),$$

where $\|\cdot\|_n$ denotes the nuclear norm, and $X = hm^*$;

- (Theoretical result): the unique minimizer high probability the true solution;
- The convex problem is expensive to solve;

¹A. Ahmed, B. Recht, and J. Romberg, Blind deconvolution using convex programming, *IEEE Transactions on Information Theory*, 60:1711-1732, 2014

Related work

- Li et al. [LLSW16]²
 - Nonconvex problem³:

Find h, m , s. t. $y = \text{diag}(Bhm^*C^*)$;

$$\min_{(h,m) \in \mathbb{C}^K \times \mathbb{C}^N} \|y - \text{diag}(Bhm^*C^*)\|_2^2;$$

²X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

³The penalty in the cost function is not added for simplicity

Related work

- Li et al. [LLSW16]²
 - Nonconvex problem³:

Find h, m , s. t. $y = \text{diag}(Bhm^*C^*)$;

$$\min_{(h,m) \in \mathbb{C}^K \times \mathbb{C}^N} \|y - \text{diag}(Bhm^*C^*)\|_2^2;$$

- (Theoretical result):
 - A good initialization
 - (Wirtinger flow method + a good initialization) $\xrightarrow{\text{high probability}}$ the true solution;

²X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

³The penalty in the cost function is not added for simplicity

Related work

- Li et al. [LLSW16]²
 - Nonconvex problem³:

Find h, m, s . t. $y = \text{diag}(Bhm^* C^*)$;

$$\min_{(h,m) \in \mathbb{C}^K \times \mathbb{C}^N} \|y - \text{diag}(Bhm^* C^*)\|_2^2;$$

- (Theoretical result):
 - A good initialization
 - (Wirtinger flow method + a good initialization) $\xrightarrow{\text{high probability}}$ the true solution;
- Lower successful recovery probability than alternating minimization algorithm empirically.

²X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

³The penalty in the cost function is not added for simplicity

Manifold Approach

Find h, m, s . t. $y = \text{diag}(Bhm^*C^*)$;

- The problem is defined on the set of rank-one matrices (denoted by $\mathbb{C}_1^{K \times N}$), neither $\mathbb{C}^{K \times N}$ nor $\mathbb{C}^K \times \mathbb{C}^N$; Why not work on the manifold directly?

Manifold Approach

Find h, m, s . t. $y = \text{diag}(Bhm^*C^*)$;

- The problem is defined on the set of rank-one matrices (denoted by $\mathbb{C}_1^{K \times N}$), neither $\mathbb{C}^{K \times N}$ nor $\mathbb{C}^K \times \mathbb{C}^N$; Why not work on the manifold directly?
- A representative Riemannian method: Riemannian steepest descent method (RSD)
 - A good initialization
 - (RSD + the good initialization) $\xrightarrow{\text{high probability}}$ the true solution;
 - The Riemannian Hessian at the true solution is well-conditioned;

Numerical Results

- Synthetic tests
 - Efficiency
 - Probability of successful recovery
- Image deblurring
 - Kernels with known supports
 - Motion kernel with unknown supports

Synthetic tests

- The matrix B is the first K column of the unitary DFT matrix;
- The matrix C is a Gaussian random matrix;
- The measurement $y = \text{diag}(Bh_{\sharp}m_{\sharp}^*C^*)$, where entries in the true solution (h_{\sharp}, m_{\sharp}) are drawn from Gaussian distribution;
- All tested algorithms use the same initial point;
- Stop when $\|y - \text{diag}(Bhm^*C^*)\|_2/\|y\|_2 \leq 10^{-8}$;

Efficiency

$$\min \|y - \text{diag}(Bhm^* C^*)\|_2^2$$

Table: Comparisons of efficiency

| | $L = 400, K = N = 50$ | | | $L = 600, K = N = 50$ | | |
|------------|-----------------------|--------------------|--------------------|-----------------------|--------------------|--------------------|
| Algorithms | [LLSW16] | [LWB13] | R-SD | [LLSW16] | [LWB13] | R-SD |
| nBh/nCm | 351 | 718 | 208 | 162 | 294 | 122 |
| $nFFT$ | 870 | 1436 | 518 | 401 | 588 | 303 |
| $RMSE$ | 2.22 ₋₈ | 3.67 ₋₈ | 2.20 ₋₈ | 1.48 ₋₈ | 2.34 ₋₈ | 1.42 ₋₈ |

- An average of 100 random runs
- nBh/nCm : the numbers of Bh and Cm multiplication operations respectively
- $nFFT$: the number of Fourier transform
- $RMSE$: the relative error $\frac{\|hm^* - h_\# m_\#^*\|_F}{\|h_\#\|_2 \|m_\#\|_2}$

[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

[LWB13]: K. Lee et. al., Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization *preprint arXiv:1312.0525*, 2013

Probability of successful recovery

- Success if $\frac{\|hm^* - h_{\#}m_{\#}^*\|_F}{\|h_{\#}\|_2\|m_{\#}\|_2} \leq 10^{-2}$

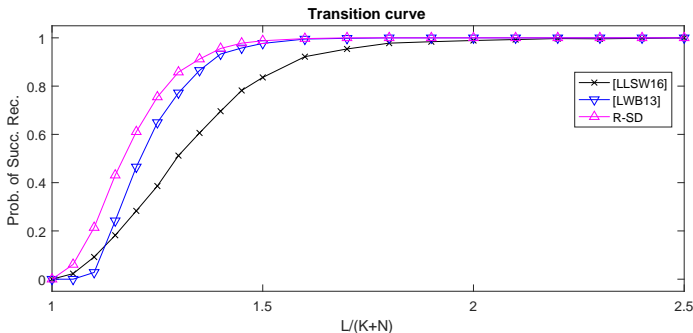


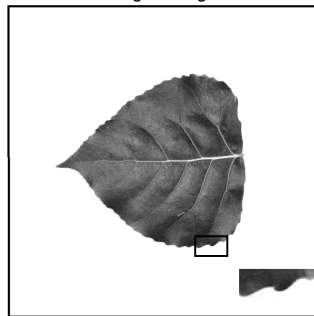
Figure: Empirical phase transition curves for 1000 random runs.

[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

[LWB13]: K. Lee et. al., Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization
preprint arXiv:1312.0525, 2013

Image deblurring

Original image



- Image [WBX⁺07]: 1024-by-1024 pixels

Image deblurring with various kernels

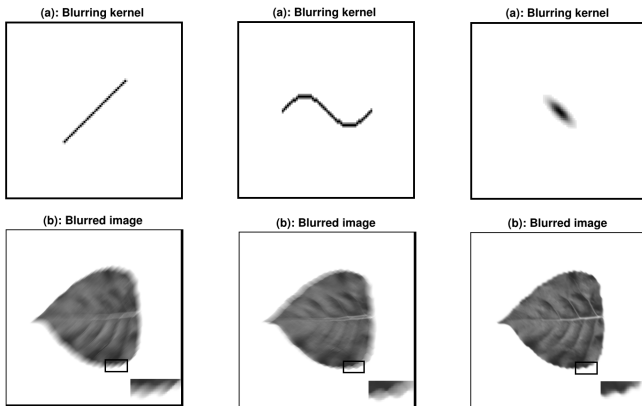


Figure: Left: Motion kernel by Matlab function “`fspecial('motion', 50, 45)`”; Middle: Kernel like function “`sin`”; Right: Gaussian kernel with covariance $[1, 0.8; 0.8, 1]$;

Image deblurring with various kernels

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis
- Support of the blurring kernel is learned from the blurred image

$$\min \|y - \text{diag}(Bhm^* C^*)\|_2^2$$

Blurred image

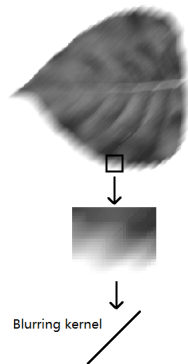


Image deblurring with various kernels

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis

Use the blurred image to learn the dominated basis vectors: **C**.

- Support of the blurring kernel is learned from the blurred image

$$\min \|y - \text{diag}(Bhm^* C^*)\|_2^2$$

Blurred image

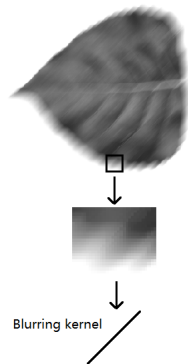


Image deblurring with various kernels

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis

Use the blurred image to learn the dominated basis vectors: **C**.

- Support of the blurring kernel is learned from the blurred image

Suppose the supports of the blurring kernels are known: **B**.

$$\min \|y - \text{diag}(Bhm^* C^*)\|_2^2$$

Blurred image

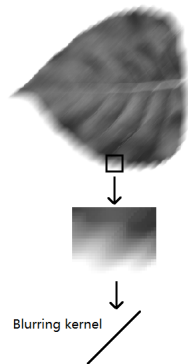


Image deblurring with various kernels

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis

Use the blurred image to learn the dominated basis vectors: **C**.

- Support of the blurring kernel is learned from the blurred image

Suppose the supports of the blurring kernels are known: **B**.

- $L = 1048576$, $N = 20000$, $K_{motion} = 109$,
 $K_{sin} = 153$, $K_{Gaussian} = 181$;

$$\min \|y - \text{diag}(Bhm^* C^*)\|_2^2$$

Blurred image

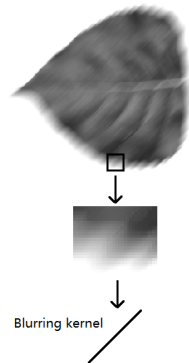


Image deblurring with various kernels

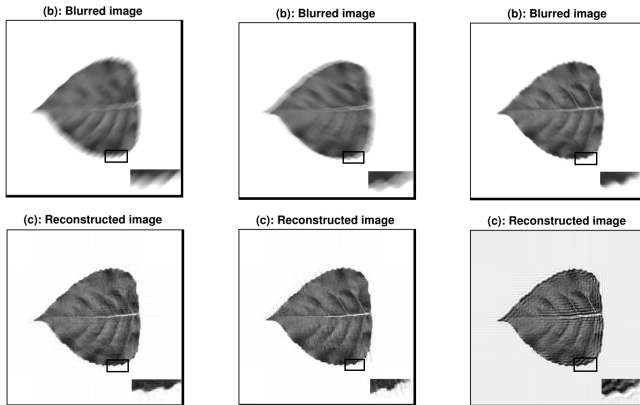


Figure: The number of iterations is 80; Computational times are about 48s;
Relative errors $\left\| \hat{\mathbf{y}} - \frac{\|\mathbf{y}\|}{\|\mathbf{y}_f\|} \mathbf{y}_f \right\| / \|\hat{\mathbf{y}}\|$ are 0.038, 0.040, and 0.089 from left to right.

○○○○○○○○○○○○○○○○
○○○○○○○○○○○○○○○○

○○○○
○○○○○○○○●○

Numerical Results

Image deblurring with unknown supports

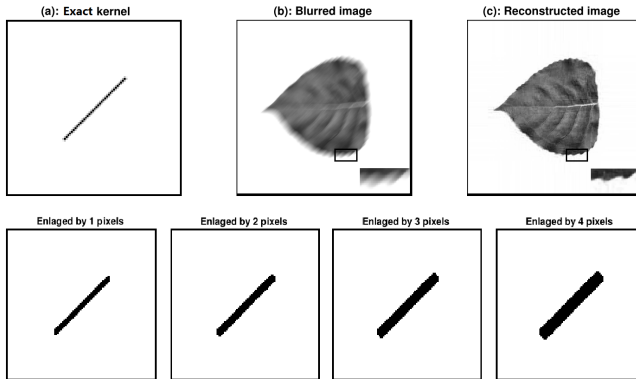


Figure: Top: reconstructed image using the exact support; Bottom: estimated supports with the numbers of nonzero entries: $K_1 = 183$, $K_2 = 265$, $K_3 = 351$, and $K_4 = 441$;

Image deblurring with unknown supports

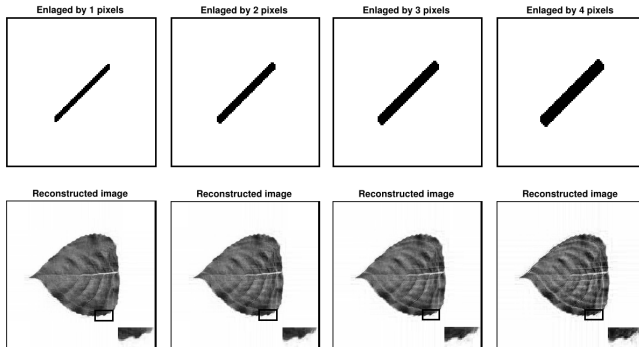


Figure: Relative errors $\left\| \hat{\mathbf{y}} - \frac{\|\mathbf{y}\|}{\|\mathbf{y}_f\|} \mathbf{y}_f \right\| / \|\hat{\mathbf{y}}\|$ are 0.044, 0.048, 0.052, and 0.067 from left to right.

Summary

- Introduced the framework of Riemannian optimization
- Used applications to show the importance of Riemannian optimization
- Briefly reviewed the history of Riemannian optimization
- Introduced the blind deconvolution problem
- Reviewed related work
- Demonstrated the performance of the Riemannian steepest descent method

Thank you

Thank you!

References I



P.-A. Absil, C. G. Baker, and K. A. Gallivan.
Trust-region methods on Riemannian manifolds.
Foundations of Computational Mathematics, 7(3):303–330, 2007.



P.-A. Absil, R. Mahony, and R. Sepulchre.
Optimization algorithms on matrix manifolds.
Princeton University Press, Princeton, NJ, 2008.



A. Ahmed, B. Recht, and J. Romberg.
Blind deconvolution using convex programming.
IEEE Transactions on Information Theory, 60(3):1711–1732, March 2014.



J. F. Cardoso and A. Souloumiac.
Blind beamforming for non-gaussian signals.
IEE Proceedings F Radar and Signal Processing, 140(6):362, 1993.



H. Drira, B. Ben Amor, A. Srivastava, M. Daoudi, and R. Slama.
3D face recognition under expressions, occlusions, and pose variations.
Pattern Analysis and Machine Intelligence, IEEE Transactions on, 35(9):2270–2283, 2013.



W. Huang, P.-A. Absil, and K. A. Gallivan.
A Riemannian symmetric rank-one trust-region method.
Mathematical Programming, 150(2):179–216, February 2015.

References II



Wen Huang, P.-A. Absil, and K. A. Gallivan.

A Riemannian BFGS Method without Differentiated Retraction for Nonconvex Optimization Problems.
SIAM Journal on Optimization, 28(1):470–495, 2018.



Wen Huang, P.-A. Absil, K. A. Gallivan, and Paul Hand.

ROPTLIB: an object-oriented C++ library for optimization on Riemannian manifolds.
 Technical Report FSU16-14, Florida State University, 2016.



W. Huang, K. A. Gallivan, and P.-A. Absil.

A Broyden Class of Quasi-Newton Methods for Riemannian Optimization.
SIAM Journal on Optimization, 25(3):1660–1685, 2015.



W. Huang, K. A. Gallivan, Anuj Srivastava, and P.-A. Absil.

Riemannian optimization for registration of curves in elastic shape analysis.
Journal of Mathematical Imaging and Vision, 54(3):320–343, 2015.
 DOI:10.1007/s10851-015-0606-8.



Wen Huang, K. A. Gallivan, and Xiangxiong Zhang.

Solving PhaseLift by low rank Riemannian optimization methods for complex semidefinite constraints.
SIAM Journal on Scientific Computing, 39(5):B840–B859, 2017.



W. Huang and P. Hand.

Blind deconvolution by a steepest descent algorithm on a quotient manifold, 2017.
 In preparation.

References III



W. Huang.

Optimization algorithms on Riemannian manifolds with applications.
 PhD thesis, Florida State University, Department of Mathematics, 2013.



H. Laga, S. Kurtek, A. Srivastava, M. Golzarian, and S. J. Miklavcic.

A Riemannian elastic metric for shape-based plant leaf classification.
2012 International Conference on Digital Image Computing Techniques and Applications (DICTA), pages 1–7, December 2012.
 doi:10.1109/DICTA.2012.6411702.



Xiaodong Li, Shuyang Ling, Thomas Strohmer, and Ke Wei.

Rapid, robust, and reliable blind deconvolution via nonconvex optimization.
CoRR, abs/1606.04933, 2016.



K. Lee, Y. Wu, and Y. Bresler.

Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization.
 pages 1–80, 2013.



Melissa Marchand, Wen Huang, Arnaud Browet, Paul Van Dooren, and Kyle A. Gallivan.

A riemannian optimization approach for role model extraction.
In Proceedings of the 22nd International Symposium on Mathematical Theory of Networks and Systems, pages 58–64, 2016.



A. Srivastava, E. Klassen, S. H. Joshi, and I. H. Jermyn.

Shape analysis of elastic curves in Euclidean spaces.
IEEE Transactions on Pattern Analysis and Machine Intelligence, 33(7):1415–1428, September 2011.
 doi:10.1109/TPAMI.2010.184.

References IV



F. J. Theis and Y. Inouye.

On the use of joint diagonalization in blind signal processing.
2006 IEEE International Symposium on Circuits and Systems, (2):7–10, 2006.



B. Vandereycken.

Low-rank matrix completion by Riemannian optimization—extended version.
SIAM Journal on Optimization, 23(2):1214–1236, 2013.



S. G. Wu, F. S. Bao, E. Y. Xu, Y.-X. Wang, Y.-F. Chang, and Q.-L. Xiang.

A leaf recognition algorithm for plant classification using probabilistic neural network.
2007 IEEE International Symposium on Signal Processing and Information Technology, pages 11–16, 2007.
[arXiv:0707.4289v1](https://arxiv.org/abs/0707.4289v1).



Xinru Yuan, Wen Huang, P.-A. Absil, and K. A. Gallivan.

A Riemannian quasi-newton method for computing the Karcher mean of symmetric positive definite matrices.
Technical Report FSU17-02, Florida State University, 2017.
www.math.fsu.edu/~whuang2/papers/RMKMSPDM.htm.