Riemannian Optimization with its Application to Blind Deconvolution Problem

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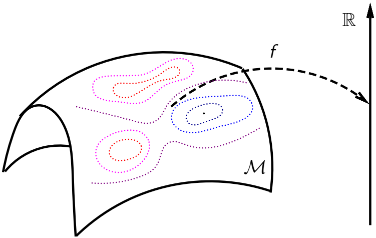
Riemannian Optimization

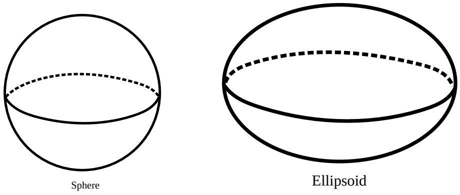
Problem: Given $f(x) : \mathcal{M} \to \mathbb{R}$, solve

$$\min_{x \in \mathcal{M}} f(x)$$

where $\mathcal{M}$ is a Riemannian manifold.



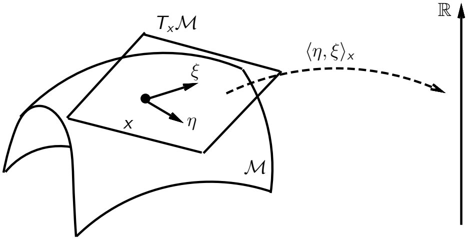
Examples of Manifolds



- Stiefel manifold: $\mathrm{St}(p, n) = \{X \in \mathbb{R}^{n \times p} | X^T X = I_p\}$
- Grassmann manifold: Set of all p-dimensional subspaces of $\mathbb{R}^n$
- Set of fixed rank m-by-n matrices
- And many more

Riemannian Manifolds

Roughly, a Riemannian manifold $\mathcal{M}$ is a smooth set with a smoothly-varying inner product on the tangent spaces.

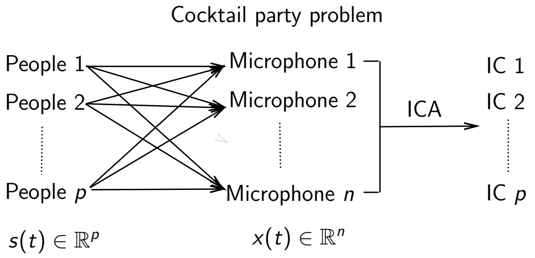


Applications

Three applications are used to demonstrate the importance of the Riemannian optimization:

- Independent component analysis [CS93]
- Matrix completion problem [Van12, HAGH16]
- Elastic shape analysis of curves [SKJJ11, HGSA15]

Application: Independent Component Analysis



- Observed signal is $x(t) = As(t)$
- One approach:
  - Assumption: $E\{s(t)s(t+\tau)\}$ is diagonal for all τ
  - $C_\tau(x) := E\{x(t)x(x+\tau)^T\} = AE\{s(t)s(t+\tau)^T\}A^T$

Application: Independent Component Analysis

- Minimize joint diagonalization cost function on the Stiefel manifold [TI06]:

$$f : \mathrm{St}(p, n) \to \mathbb{R} : V \mapsto \sum_{i=1}^{N} \|V^T C_i V - \mathrm{diag}(V^T C_i V)\|_F^2.$$

- $C_1, \dots, C_N$ are covariance matrices and $\mathrm{St}(p, n) = \{X \in \mathbb{R}^{n \times p} | X^T X = I_p\}$.

Application: Matrix Completion Problem

Matrix completion problem

	Movie 1	Movie 2				Movie n
User 1		1			4	
User 2	3	5			4	
			5	1		
User m		2			5	3

Rate matrix M

- The matrix M is sparse
- The goal: complete the matrix M

Application: Matrix Completion Problem

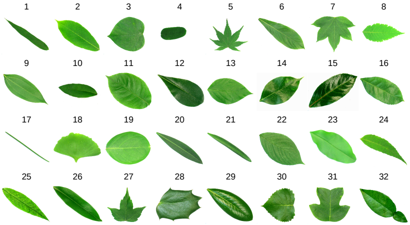
$$\begin{array}{c}\text{movies}\\ \begin{pmatrix} a_{11} & & & a_{14} \\ & & & a_{24} \\ & & a_{33} & \\ a_{41} & & & \\ & a_{52} & a_{53} & \end{pmatrix}\end{array} = \begin{array}{c}\text{meta-user}\\ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{pmatrix}\end{array} \begin{array}{c}\text{meta-movie}\\ \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix}\end{array}$$

- Minimize the cost function

$$f : \mathbb{R}_r^{m\times n} \to \mathbb{R} : X \mapsto f(X) = \|P_\Omega M - P_\Omega X\|_F^2.$$

- $\mathbb{R}_r^{m\times n}$ is the set of m-by-n matrices with rank r. It is known to be a Riemannian manifold.

Application: Elastic Shape Analysis of Curves



- Classification [LKS+12, HGSA15]
- Face recognition [DBS+13]

Application: Elastic Shape Analysis of Curves

- Elastic shape analysis invariants:
  - Rescaling
  - Translation
  - Rotation
  - Reparametrization
- The shape space is a quotient space

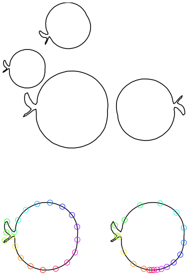
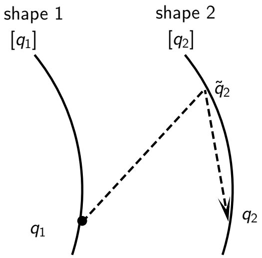


Figure: All are the same shape.

Application: Elastic Shape Analysis of Curves



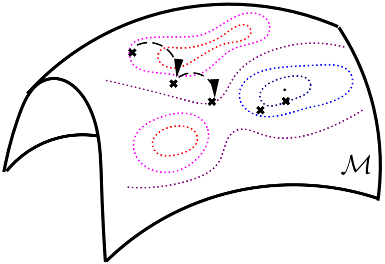
- Optimization problem $\min_{q_2 \in [q_2]} \mathrm{dist}(q_1, q_2)$ is defined on a Riemannian manifold
- Computation of a geodesic between two shapes
- Computation of Karcher mean of a population of shapes

More Applications

- Matrix/tensor completion [HAGH16]
- Role model extraction [MHB^{+}16]
- Computations on SPD matrices [YHAG17]
- Elastic shape analysis [HGSA15, YHGA15, HYGA15]
- Phase retrieval problem [HGZ16]
- Blind deconvolution [HH17]
- Synchronization of rotations [Hua13]
- Low-rank approximate solution for Lyapunov equation

Comparison with Constrained Optimization

- All iterates on the manifold
- Convergence properties of unconstrained optimization algorithms
- No need to consider Lagrange multipliers or penalty functions
- Exploit the structure of the constrained set



Iterations on the Manifold

Consider the following generic update for an iterative Euclidean optimization algorithm:

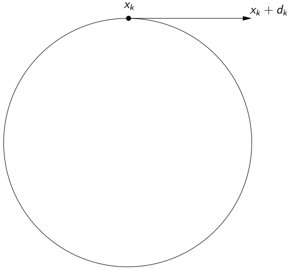
$$x_{k+1} = x_k + \Delta x_k = x_k + \alpha_k s_k \ .$$

This iteration is implemented in numerous ways, e.g.:

- Steepest descent: $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$
- Newton's method: $x_{k+1} = x_k - \left[\nabla^2 f(x_k)\right]^{-1} \nabla f(x_k)$
- Trust region method: Δx_k is set by optimizing a local model.

Riemannian Manifolds Provide

- Riemannian concepts describing directions and movement on the manifold
- Riemannian analogues for gradient and Hessian



Riemannian gradient and Riemannian Hessian

Definition

The Riemannian gradient of f at x is the unique tangent vector in $T_x M$ satisfying $\forall \eta \in T_x M$, the directional derivative

$$\mathrm{D}\, f(x)[\eta] = \langle \mathrm{grad}\, f(x), \eta \rangle$$

and $\mathrm{grad}\, f(x)$ is the direction of steepest ascent.

Definition

The Riemannian Hessian of f at x is a symmetric linear operator from $T_x M$ to $T_x M$ defined as

$$\mathrm{Hess}\, f(x) : T_x M \to T_x M : \eta \to \nabla_\eta \,\mathrm{grad}\, f,$$

where ∇ is the affine connection.

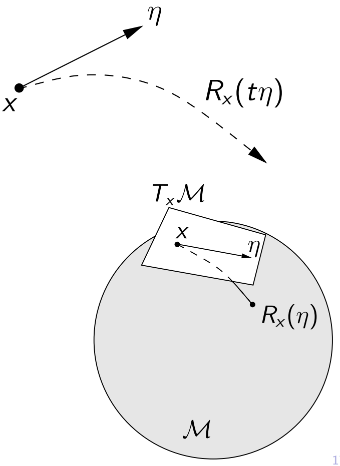
Retractions

Euclidean	Riemannian
$x_{k+1} = x_k + \alpha_k d_k$	$x_{k+1} = R_{x_k}(\alpha_k \eta_k)$

Definition

A retraction is a mapping R from TM to M satisfying the following:

- R is continuously differentiable
- $R_x(0) = x$
- $\mathrm{D}\, R_x(0)[\eta] = \eta$



- maps tangent vectors back to the manifold
- defines curves in a direction

Categories of Riemannian optimization methods

Retraction-based: local information only

Line search-based: use local tangent vector and $R_x(t\eta)$ to define line

- Steepest decent
- Newton

Local model-based: series of flat space problems

- Riemannian trust region Newton (RTR)
- Riemannian adaptive cubic overestimation (RACO)

Categories of Riemannian optimization methods

Retraction and transport-based: information from multiple tangent spaces

- Conjugate gradient: multiple tangent vectors
- Quasi-Newton e.g. Riemannian BFGS: transport operators between tangent spaces

Additional element required for optimizing a cost function (M, g):

- formulas for combining information from multiple tangent spaces.

Vector Transports

Vector Transport

- Vector transport: Transport a tangent vector from one tangent space to another
- $\mathcal{T}_{\eta_x}\xi_x$, denotes transport of ξ_x to tangent space of $R_x(\eta_x)$. R is a retraction associated with $\mathcal{T}$

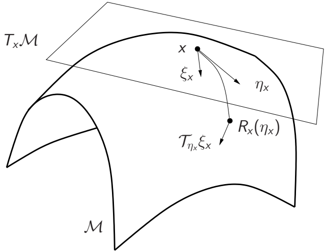


Figure: Vector transport.

Retraction/Transport-based Riemannian Optimization

Benefits

- Increased generality does not compromise the important theory
- Less expensive than or similar to previous approaches
- May provide theory to explain behavior of algorithms specifically developed for a particular application – or closely related ones

Possible Problems

- May be inefficient compared to algorithms that exploit application details

Some History of Optimization On Manifolds (I)

Luenberger (1973), *Introduction to linear and nonlinear programming*. Luenberger mentions the idea of performing line search along geodesics, "which we would use if it were computationally feasible (which it definitely is not)". Rosen (1961) essentially anticipated this but was not explicit in his Gradient Projection Algorithm.

Gabay (1982), *Minimizing a differentiable function over a differential manifold*. Steepest descent along geodesics; Newton's method along geodesics; Quasi-Newton methods along geodesics. On Riemannian submanifolds of $\mathbb{R}^n$.

Smith (1993-94), *Optimization techniques on Riemannian manifolds*. Levi-Civita connection ∇; Riemannian exponential mapping; parallel translation.

Some History of Optimization On Manifolds (II)

The "pragmatic era" begins:

Manton (2002), *Optimization algorithms exploiting unitary constraints* "The present paper breaks with tradition by not moving along geodesics". The geodesic update $\mathrm{Exp}_x \eta$ is replaced by a projective update $\pi(x + \eta)$, the *projection* of the point $x + \eta$ onto the manifold.

Adler, Dedieu, Shub, et al. (2002), *Newton's method on Riemannian manifolds and a geometric model for the human spine*. The exponential update is relaxed to the general notion of *retraction*. The geodesic can be replaced by any (smoothly prescribed) curve tangent to the search direction.

Absil, Mahony, Sepulchre (2007) Nonlinear conjugate gradient using retractions.

Some History of Optimization On Manifolds (III)

Theory, efficiency, and library design improve dramatically:

Absil, Baker, Gallivan (2004-07), Theory and implementations of Riemannian Trust Region method. Retraction-based approach. Matrix manifold problems, software repository

http://www.math.fsu.edu/~cbaker/GenRTR

Anasazi Eigenproblem package in Trilinos Library at Sandia National Laboratory

Absil, Gallivan, Qi (2007-10), Basic theory and implementations of Riemannian BFGS and Riemannian Adaptive Cubic Overestimation. Parallel translation and Exponential map theory, Retraction and vector transport empirical evidence.

Some History of Optimization On Manifolds (IV)

Ring and With (2012), combination of differentiated retraction and isometric vector transport for convergence analysis of RBFGS

Absil, Gallivan, Huang (2009-2017), Complete theory of Riemannian Quasi-Newton and related transport/retraction conditions, Riemannian SR1 with trust-region, RBFGS on partly smooth problems, A C++ library: `http://www.math.fsu.edu/~whuang2/ROPTLIB`

Sato, Iwai (2013-2015), Zhu (2017), Global convergence analysis for Riemannian conjugate gradient methods

Bonnabel (2011), Sato, Kasai, Mishra(2017) Riemannian stochastic gradient descent method.

Many people Application interests increase noticeably

Current UCL/FSU Methods

- Riemannian Steepest Descent [AMS08]
- Riemannian conjugate gradient [AMS08]
- Riemannian Trust Region Newton [ABG07]: global, quadratic convergence
- Riemannian Broyden Family [HGA15, HAG18] : global (convex), superlinear convergence
- Riemannian Trust Region SR1 [HAG15]: global, $(d+1)-$superlinear convergence
- For large problems
  - Limited memory RTRSR1
  - Limited memory RBFGS

Current UCL/FSU Methods

Riemannian manifold optimization library (ROPTLIB) is used to optimize a function on a manifold.

- Most state-of-the-art methods;
- Commonly-encountered manifolds;
- Written in C++;
- Interfaces with Matlab, Julia and R;
- BLAS and LAPACK;
- `www.math.fsu.edu/~whuang2/Indices/index_ROPTLIB.html`

Current/Future Work on Riemannian methods

- Manifold and inequality constraints
- Discretization of infinite dimensional manifolds and the convergence/accuracy of the approximate minimizers – specific to a problem and extracting general conclusions
- Partly smooth cost functions on Riemannian manifold
- Limited-memory quasi-Newton methods on manifolds

Blind deconvolution

[Blind deconvolution]

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Blurred image

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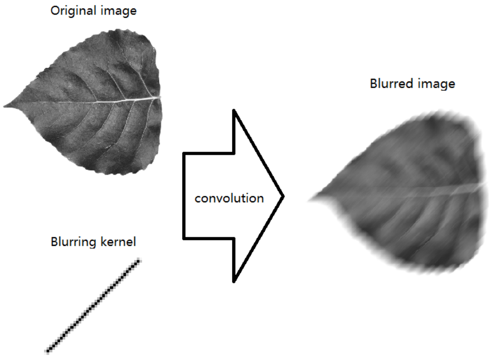

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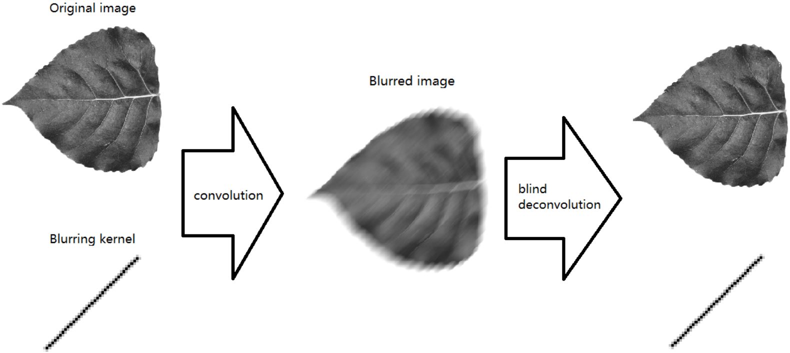

Problem Statement

[Blind deconvolution (Discretized version)]

Blind deconvolution is to recover two unknown signals $\mathbf{w} \in \mathbb{C}^L$ and $\mathbf{x} \in \mathbb{C}^L$ from their convolution $\mathbf{y} = \mathbf{w} * \mathbf{x} \in \mathbb{C}^L$.

- We only consider circular convolution:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \vdots \\ \mathbf{y}_L \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_L & \mathbf{w}_{L-1} & \dots & \mathbf{w}_2 \\ \mathbf{w}_2 & \mathbf{w}_1 & \mathbf{w}_L & \dots & \mathbf{w}_3 \\ \mathbf{w}_3 & \mathbf{w}_2 & \mathbf{w}_1 & \dots & \mathbf{w}_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_L & \mathbf{w}_{L-1} & \mathbf{w}_{L-2} & \dots & \mathbf{w}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \mathbf{x}_L \end{bmatrix}$$

- Let $y = \mathbf{F}\mathbf{y}$, $w = \mathbf{F}\mathbf{w}$, and $x = \mathbf{F}\mathbf{x}$, where $\mathbf{F}$ is the DFT matrix;
- $y = w \odot x$, where $\odot$ is the Hadamard product, i.e., $y_i = w_i x_i$.
- Equivalent question: Given y, find w and x.

Problem Statement

Problem: Given $y \in \mathbb{C}^L$, find $w, x \in \mathbb{C}^L$ so that $y = w \odot x$.

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- Assumption: w and x are in known subspaces, i.e., $w = Bh$ and $x = \overline{Cm}$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$;

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  - Reasonable in various applications;
  - Leads to mathematical rigor; $(L/(K + N)$ reasonably large)

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Problem under the assumption

Given $y \in \mathbb{C}^L$, $B \in \mathbb{C}^{L \times K}$ and $C \in \mathbb{C}^{L \times N}$, find $h \in \mathbb{C}^K$ and $m \in \mathbb{C}^N$ so that

$$y = Bh \odot \overline{Cm} = \mathrm{diag}(Bhm^* C^*).$$

Related work

Find h, m, s. t. $y = \mathrm{diag}(Bhm^* C^*)$;

- Ahmed et al. [ARR14][1]
  - Convex problem:

$$\min_{X \in \mathbb{C}^{K \times N}} \|X\|_n, \text{ s. t. } y = \mathrm{diag}(BXC^*),$$

where $\|\cdot\|_n$ denotes the nuclear norm, and $X = hm^*$;

[1] A. Ahmed, B. Recht, and J. Romberg, Blind deconvolution using convex programming, *IEEE Transactions on Information Theory*, 60:1711-1732, 2014

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    where $\|\cdot\|_n$ denotes the nuclear norm, and $X = hm^*$;
  - (Theoretical result): the unique minimizer $\xrightarrow{\text{high probability}}$ the true solution;
  - The convex problem is expensive to solve;

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Related work

$$\boxed{\text{Find } h, m, \text{ s. t. } y = \mathrm{diag}(Bhm^* C^*);}$$

- Li et al. [LLSW16][2]
  - Nonconvex problem[3]:

$$\min_{(h,m)\in\mathbb{C}^K\times\mathbb{C}^N} \|y - \mathrm{diag}(Bhm^* C^*)\|_2^2;$$

[2] X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

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  - (Theoretical result):
    - A good initialization
    - (Wirtinger flow method + a good initialization) $\xrightarrow{\text{high probability}}$ the true solution;

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  - (Theoretical result):
    - A good initialization
    - (Wirtinger flow method + a good initialization) $\xrightarrow{\text{high probability}}$ the true solution;
  - Lower successful recovery probability than alternating minimization algorithm empirically.

[2] X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016

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Manifold Approach

Find h, m, s. t. $y = \mathrm{diag}(Bhm^* C^*)$;

- The problem is defined on the set of rank-one matrices (denoted by $\mathbb{C}_1^{K \times N}$), neither $\mathbb{C}^{K \times N}$ nor $\mathbb{C}^K \times \mathbb{C}^N$; Why not work on the manifold directly?

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- A representative Riemannian method: Riemannian steepest descent method (RSD)
  - A good initialization
  - (RSD + the good initialization) $\xrightarrow{\text{high probability}}$ the true solution;
  - The Riemannian Hessian at the true solution is well-conditioned;

Efficiency

Table: Comparisons of efficiency

	$L = 400, K = N = 50$			$L = 600, K = N = 50$		
Algorithms	[LLSW16]	[LWB13]	R-SD	[LLSW16]	[LWB13]	R-SD
nBh/nCm	351	718	208	162	294	122
nFFT	870	1436	518	401	588	303
RMSE	2.22_{-8}	3.67_{-8}	2.20_{-8}	1.48_{-8}	2.34_{-8}	1.42_{-8}

- An average of 100 random runs
- nBh/nCm: the numbers of Bh and Cm multiplication operations respectively
- nFFT: the number of Fourier transform
- RMSE: the relative error $\frac{\|hm^* - h_\sharp m_\sharp^*\|_F}{\|h_\sharp\|_2 \|m_\sharp\|_2}$

[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016
[LWB13]: K. Lee et. al., Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization *preprint arXiv:1312.0525*, 2013

Probability of successful recovery

- Success if $\frac{\|hm^* - h_\sharp m_\sharp^*\|_F}{\|h_\sharp\|_2 \|m_\sharp\|_2} \leq 10^{-2}$

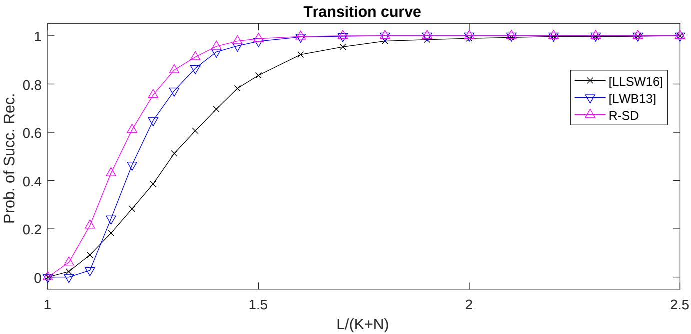
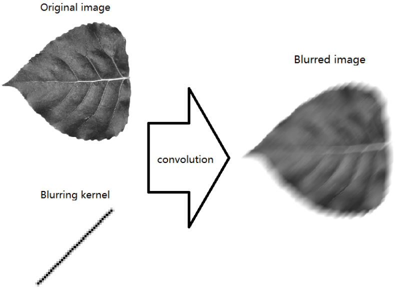


Figure: Empirical phase transition curves for 1000 random runs.

[LLSW16]: X. Li et. al., Rapid, robust, and reliable blind deconvolution via nonconvex optimization, *preprint arXiv:1606.04933*, 2016
[LWB13]: K. Lee et. al., Near Optimal Compressed Sensing of a Class of Sparse Low-Rank Matrices via Sparse Power Factorization *preprint arXiv:1312.0525*, 2013

Image deblurring

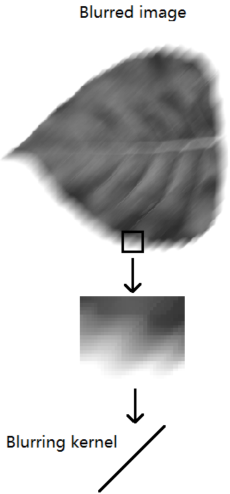


- Original image [WBX^{+}07]: 1024-by-1024 pixels
- Motion blurring kernel (Matlab: fspecial('motion', 50, 45))

Image deblurring

What subspaces are the two unknown signals in?

- Image is approximately sparse in the Haar wavelet basis
- Support of the blurring kernel is learned from the blurred image



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  Use the blurred image to learn the dominated basis: **C**.

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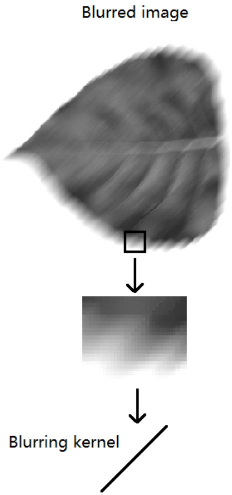

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  Use the blurred image to learn the dominated basis: **C**.

- Support of the blurring kernel is learned from the blurred image

  Suppose the support of the blurring kernel is known: **B**.

- $L = 1048576, K = 109,$
  $N = 5000, 20000, 80000$

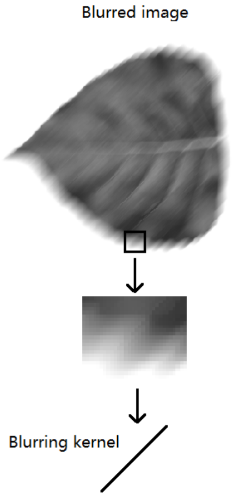

Image deblurring

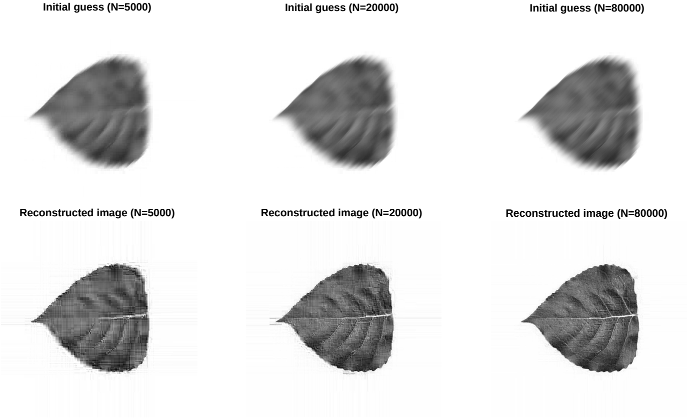

Figure: Initial guess by running power method for 50 iterations and the reconstructed image for $N = 5000, 20000$, and 80000. Computational time: 2-3 mins.

Summary

- Introduced the framework of Riemannian optimization
- Used applications to show the importance of Riemannian optimization
- Introduced the blind deconvolution problem
- Showed the performance of the Riemannian steepest descent method in this application

Thank you

Thank you!

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