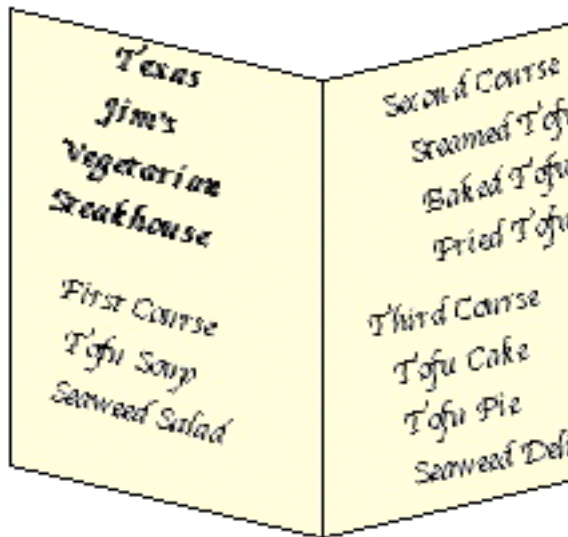


UNIT 3 MODULE 1

THE FUNDAMENTAL COUNTING PRINCIPLE

EXAMPLE 3.1.1



Plato is going to choose a three-course meal at his favorite restaurant. He must choose one item from each of the following three categories.

First course: Tofu Soup (TS); Seaweed Salad (SS)

Second course: Steamed Tofu (ST); Baked Tofu (BT); Fried Tofu (FT);

Third course: Tofu Cake (TC); Tofu Pie (TP); Seaweed Delight (SD)

How many different three-course meals are possible?

Solve this problem by listing every possible 3-course meal.

SOLUTION

We will list every possible 3-course meal:

- | | | | |
|--------------|--------------|--------------|--------------|
| 1. TS-ST-TC | 2. TS-ST-TP | 3. TS-ST-SD | 4. TS-BT-TC |
| 5. TS-BT-TP | 6. TS-BT-SD | 7. TS-FT-TC | 8. TS-FT-TP |
| 9. TS-FT-SD | 10. SS-ST-TC | 11. SS-ST-TP | 12. SS-ST-SD |
| 13. SS-BT-TC | 14. SS-BT-TP | 15. SS-BT-SD | 16. SS-FT-TC |
| 17. SS-FT-TP | 18. SS-FT-SD | | |

We see that there are 18 different three-course meals.

However, there is an easier way to get at this answer. Notice that in order to choose a three-course meal, we need to make three decisions:

1. Choose item for first course: There are 2 options.
2. Choose item for second course: There are 3 options
3. Choose option for third course: There are 3 options.

Now observe that $(2)(3)(3) = 18$.

This is not a coincidence. It is an illustration of the **Fundamental Counting Principle**.

The Fundamental Counting Principle (FCP)

To determine the number of different outcomes possible in some complex process:

1. Analytically break down the process into separate stages or decisions.
2. Count the number of **options** that are available at each stage or decision.
3. Multiply together all of the numbers from Step 2 above.

UNIT 3 MODULE 1

EXAMPLE 3.1.2

How many different three course meals are possible if Plato has already decided that he won't have Baked Tofu and he will have Seaweed Delight?

SOLUTION

We will use the Fundamental Counting Principle.

1. Choose item for first course: 2 options
 2. Choose item for second course: 2 options (since he won't choose Baked Tofu)
 3. Choose item for third course: 1 option (since it has already been decided that he will choose Seaweed Delight).
- $$(2)(2)(1) = 4 \text{ different 3-course meals}$$

The primary advantage to the use of the Fundamental Counting Principle becomes apparent when we have more complicated decision processes. For example:

EXAMPLE 3.1.3

Referring to the situation in the previous example, suppose that the restaurant becomes quite successful and so the operators decide to expand their menu. Now there are 5 items for the first course, 7 items for the second course, 4 items for the third course, and a new fourth course is added (the seltzer course, in which we choose between *Bromo* and *Alka*). How many four-course meals are possible?

SOLUTION

To choose a 4-course meal requires that we make four decisions. There are 5 options, 7 options, 4 options and 2 options, respectively, for each of these decisions.

$$(5)(7)(4)(2) = 280 \text{ different 4-course meals.}$$

It would be very cumbersome to try to solve this problem by listing every possible outcome, since there are so many different outcomes.

EXAMPLE 3.1.4

1. A student will schedule her classes next semester by choosing one course from each of the following categories:

- i. ARH3130, ARH3150, or ARH4110
- ii. STA1013, CGS2030, MGF1107 or MAC1105.
- iii. ENC1142, ENC1144, or ENC1145
- iv. WOH1023, WOH1030, AMH1000, EUH2100 or AFH1000.

How many different 4-course combinations are possible?

- A. 180
- B. 27
- C. 15
- D. 16

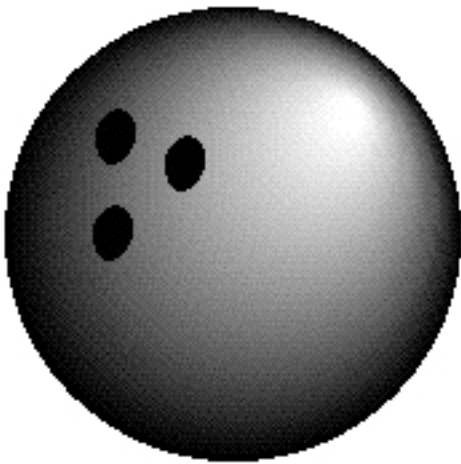
2. How many 4-course combinations are possible if she knows that she can't take ARH4110 and she will take STA1013?

UNIT 3 MODULE 1

EXAMPLE 3.1.5

Prior to the coin toss for a football game, the captains of the two teams meet at midfield. Team A has 4 captains, and team B has 3 captains. Each captain of team A shakes hands once with each captain of team B. Bill Gates has recently acquired the patent on handshakes, so he wants to know how many handshakes occur, in order to collect his royalty. How many handshakes occur?

EXAMPLE 3.1.6



There are 5 guys (including Gomer) on Gomer's bowling team. After the beer frame they will each choose one of the following: *Scud*, *Scud Lite*, or *Scud Ice*. How many outcomes are possible?

- A. 60
- B. 125
- C. 15
- D. 243
- E. 120

EXAMPLE 3.1.7

In Florida, standard automobile license plate "numbers" used to follow the following scheme:
3 letters -- 2 digits -- 1 letter

Examples: JKP47R

TRR39S

VWN22Y

ZQW05Z

How many different license plate codes were possible under this scheme?

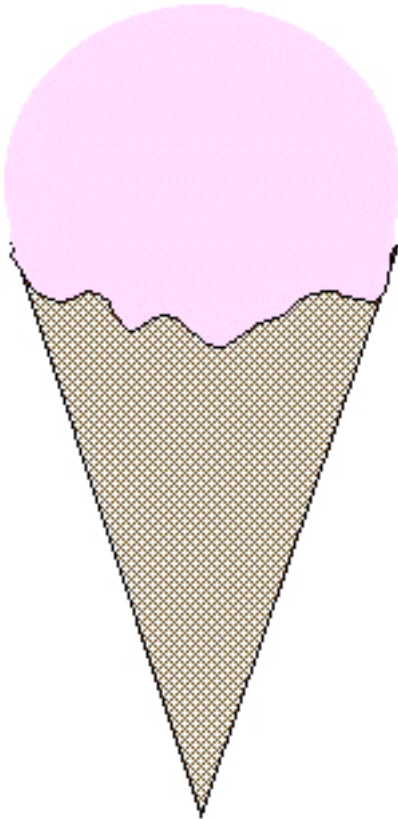
EXAMPLE 3.1.8

Gomer has to take a 5 question true/false quiz, but he hasn't studied. He will guess at each problem. In how many different ways is it possible to answer the quiz questions? How likely is it that he will get a score of 100%?

EXAMPLE 3.1.9

Gomer has to take a 25 question multiple-choice test, but he hasn't studied. He will guess at each problem. For each problem the possible responses are A, B, C, or D. In how many different ways is it possible to answer the test questions? How likely is it that he will get a score of 100%?

EXAMPLE 3.1.10



Gomer is going to order a frozen tofu cone from *I Definitely Believe It's Tofu*.

The following toppings are available:

1. carob chips
2. frosted alfalfa sprouts
3. seaweed sprinkles
4. rolled oats
5. rose hips

He may choose all, some or none of these toppings.
How many topping combinations are possible?

- A. 5
- B. 10
- C. 25
- D. 32
- E. 120

UNIT 3 MODULE 1

EXAMPLE 3.1.11

1. How many different 4-digit numbers can be formed using the following digits? (Note: the first digit cannot be 0, or else the number would be a 3-digit number).
 $\{0, 2, 3, 5, 8\}$
2. How many different 4-digit numbers that are multiples of 5 can we form?

EXAMPLE 3.1.12

Gomer is considering the purchase of a new super-cheap sport/utility vehicle, the Skuzuzi Kamikaze. He must choose a vehicle, taking into account the following options:

- i. **Transmission:** 4-speed standard transmission, 5-speed standard transmission, or automatic transmission;
- ii. **Bumper:** steel bumpers, vinyl bumpers or 2x4 boards bolted to the front and back;
- iii. **Top:** hard-top, vinyl top convertible, or chicken wire stapled over the roll bar;
- iv. **Funerary accessory:** complementary funeral wreath or cremation urn.

How many different vehicle option packages are possible?

EXAMPLE 3.1.13

Homer, Gomer, Plato, Euclid, Socrates, Aristotle, Homerina and Gomerina form the board of directors of the Lawyer and Poodle Admirers Club. They will choose from amongst themselves a Chairperson, Secretary, and Treasurer. No person will hold more than one position. How many different outcomes are possible?

- A. 336 B. 24 C. 512 D. 21

EXAMPLE 3.1.13 SOLUTION

Choosing a Chairperson, Secretary and Treasurer from among these 8 people requires us to make three decisions. However (unlike in all of the previous examples) these three decisions are **not independent**. For instance, the choice we make when we select the chairperson affects which options are available when we go to choose the Secretary, since the person selected to be Chairperson cannot also be selected to be Secretary.

- i. Choose Chairperson: 8 options;
- ii. Choose Secretary: 7 options (one person has already been chosen to be Chairperson);
- iii. Choose Treasurer: 6 options (two people have already been chosen to be Chairperson and Secretary, respectively).

According to the Fundamental Counting Principle the number of outcomes is:

$$(8)(7)(6) = 336.$$

UNIT 3 MODULE 1

SPECIAL TERMINOLOGY In the previous example, we determined that there were 336 different ways to choose 3 people from a set of 8 people, and arrange the 3 people according to who gets which title. In general, the number of ways to choose and arrange 3 objects from a set of 8 objects is called "**the number of permutations of 8 things taken 3 at a time,**" and this number is always 336.

More special terminology: we say that a series of decisions are **independent** if the result of one decision does not affect the options that are available for other decisions. On the other hand, if the result of one decision limits the options that are available for other decisions, we say that the decisions are **dependent**.

EXAMPLE 3.1.14

A couple is expecting the birth of a baby boy. They will name the child by choosing a first name and middle name from this list of their favorite boys' names: Jacob, Jonah, Joshua, Jeremy, James, Joseph. The first name will be different from the middle name. How many different two-part names are possible? (Example: Joseph Jeremy, Jeremy Joseph, and Joshua James are three different, valid two-part names; Jacob Jacob is not a valid possibility.)

- A. 36 B. 12 C. 30 D. 11

Terminology: in the previous example we saw that there are 30 ways to choose two different names from a list of six names, and arrange the two names according to which one is the "first name" and which one is the "middle name." More generally, the number of ways to choose and arrange 2 objects from a set of 6 objects is called **the number of permutations of 6 things taken 2 at a time,** and this number will always be 30.

EXAMPLE 3.1.15

Erasmus is trying to guess the combination to his combination lock. The "combination" is a sequence of three numbers, where the numbers range from 1 to 12, with no numbers repeated. How many different "combinations" are possible if he knows that the last number in the combination is either 1 or 11?

- A. 264 B. 1320 C. 220 D. 288

EXAMPLE 3.1.15 solution

Choosing a three-number "combination" having no repeated numbers requires that we make three dependent decisions. One of these decisions, however, has a special condition attached to it (the third number must be either 1 or 11). **When using the Fundamental Counting Principle in a situation involving dependent decisions, if one decision has a special condition, that decision must be treated first,** because the special condition overrides the other decisions. For example, that fact that the third number must be 1 or 11 means that it is impossible for the "combination" to simultaneously have 1 for the first number and 11 for the second number (since then there would be nothing left for the third number).

UNIT 3 MODULE 1

Three dependent decisions:

1. Choose **third** number (two options);
2. Choose first number (11 options);
3. Choose second number (10 options).

According to the Fundamental Counting Principle the number of possible outcomes is $(2)(11)(10) = 220$.

The correct choice is C.

EXAMPLE 3.1.16

Adam, Beth, Charles, Dawn, and Ernie are the five finalists in the fabulous Clearers Publishinghouse sweepstakes. Through a random selection process, one of them will win a pen-and-pencil set, one of them will win a one-year supply of cat food, and one of them will win a brand new Chevy blazer (that is, a vinyl sports jacket with the words "Chevy" embossed on the back). Nobody will win more than one prize.

How many different outcomes are possible?

- A. 125 B. 15 C. 12 D. 60

EXAMPLE 3.1.17

The Shakesperean Insult-O-Matic

A	B
dreadful	fiend
unmanner'd	dog
murderous	hog
wrangling	cacodemon
elvish-mark'd	dissembler
abortive	toad
rooting	homicide
foul	hedgehog
defused	infection of a man
cursed	villain
devilish	lump of foul deformity
detested	devil
poisonous	minister of hell
bunchback'd	beggar
	bottled spider
	slander of thy mother's heavy womb

Suppose we want to form a three-part Shakespearean insult, such as "Thou **dreadful, wrangling, minister of hell**" by choosing two adjectives from column A and one noun phrase

UNIT 3 MODULE 1

from column B. How many different insults are possible? (Assume, for instance, that "Unmanner'd, elvish-marked, lump of foul deformity" is different from "Elvish-marked, unmanner'd, lump of foul deformity.")

The vocabulary comes from Act I of *King Richard III*.

EXAMPLE 3.1.18

Homer is going to buy a 1986 Yugo. He has shopped around, and has found that '86 Yugos are available with any combination of the following options: plush Corinthian vinyl interior, padded steering wheel, tinted windows, bucket seats, motor, windshield. How many different option packages are possible?

EXAMPLE 3.1.20

The Egotists' Club has 6 members: A, B, C, D, E, and F. They are going to line up, from left to right, for a group photo. After lining up in alphabetical order (ABCDEF), Mr. F complains that he is always last whenever they do things alphabetically, so they agree to line up in reverse order (FEDCBA) and take another picture. Then Ms. D complains that she's always stuck next to Mr. C, and that she never gets to be first in line. Finally, in order to avoid bruised egos, they all agree to take pictures for every possible left-to-right line-up of the six people. How many different photos must be taken?

SOLUTION

To arrange the six people in a line we need to make six dependent decisions:

1. Choose first person (6 options);
2. Choose second person (5 options)
3. Choose third person (4 options)
4. Choose fourth person (3 options)
5. Choose fifth person (2 options)
6. Choose last person (1 option)

According to the Fundamental Counting Principle the number of outcomes is $(6)(5)(4)(3)(2)(1) = 720$.

Note that the number of ways to arrange 6 people is equal to 6 multiplied all of the positive integers smaller than 6. Likewise it is easy to see that if there were 8 people rather than 6 people, the number of ways to arrange them in a line would be

$(8)(7)(6)(5)(4)(3)(2)(1)$

and if there were 10 people instead of 6 or 8 the number of ways to arrange them in a line would be $(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)$

These products are called **factorials**.

$(6)(5)(4)(3)(2)(1)$ is called "6 factorial" and is denoted **6!**

$(8)(7)(6)(5)(4)(3)(2)(1)$ is called "8 factorial" and is denoted **8!**

$(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)$ is called "10 factorial" and is denoted **10!**

UNIT 3 MODULE 1

In general, if n is a positive integer, then “ n factorial” (denoted $n!$) is the product of n multiplied by all of the positive integers smaller than n :

$$n! = (n)(n-1)(n-2)\dots(3)(2)(1)$$

Factorials are important in the theory of counting because $n!$ is the number of ways to arrange n objects.

WORLD WIDE WEB NOTE

For practice problems involving the Fundamental Counting Principle visit the companion web site and try THE FUNDAMENTALIZER. (Some of those problems require the methods of Unit 3 Module 2, however.)

HOMEWORK EXERCISES

1. A 'combination' lock has a dial bearing the numbers 1 through 16. How many different 3-number 'combinations' are possible if there are no restrictions on the 3 numbers (example: 16-5-4, 5-16-4, 3-3-12 are three different, valid 'combinations').
A. 48 B. 4096 C. 3360 D. 256
2. Prior to the coin toss for a football game, the captains for the two teams meet at midfield. One team has three captains and the other team has five captains. Each captain from the first team shakes hands with each captain from the other team. How many handshakes occur?
A. 30 B. 8 C. 15 D. 17
3. A seven-question quiz has four true/false questions followed by 3 multiple choice questions. For each multiple choice question there are four possible answers. In how many different ways is it possible to answer the seven questions?
A. 12 B. 80 C. 28 D. 1024
4. Socrates is going to buy a new motorcycle. The motorcycles that interest him come with the following optional accessories: extra-noisy pipes, sidecar, trailer, training wheels. How many different accessory combinations are possible?
A. 16 B. 8 C. 24 D. 10
5. When Euclid dresses up for goth night, he has to choose a cloak, a shade of dark lipstick, and a pair of boots. He has two cloaks, 6 shades of dark lipstick, and 3 pairs of boots. How many different combinations are possible?
A. 15 B. 24 C. 11 D. 36
6. Harpo, Groucho, Chico, Zeppo, Gummo and Karl are running in a race. How many different orders of finish are possible?
A. 36 B. 64 C. 720 D. 46656
7. Socrates, Euclid, Plato, Aristotle, Diogenes, and Democritus are going to choose from amongst themselves one person to wax the floor and another person to scrub the toilet. How many different outcomes are possible if Diogenes will not scrub the toilet?
A. 36 B. 720 C. 30 D. 25

UNIT 3 MODULE 1

8. Gomer is going to order a pizza for his cat, Fido. The following toppings are available for cat pizzas: sparrow sprinkles, lizard lozenges, mousie munchies, butterfly bits, and beetle bites. He may choose any combination of those toppings (this includes the possibility that he may choose none of them). How many different topping combinations are possible?

- A. 32 B. 25 C. 120 D. 3125

9. When Diogenes the used-car salesman gets dressed for work, he has to choose a jacket, shirt, tie, belt, trousers and shoes. He has a plaid jacket, a checkered jacket, and a pink jacket; a white long-sleeved shirt, a white short-sleeved shirt, a green short-sleeved shirt and a yellow long-sleeved shirt; a striped tie, a polka-dotted tie, a checkered tie, and a solid orange tie; a white patent-leather belt and a black patent-leather belt; black trousers, brown trousers, gray trousers, plaid trousers and pin-striped trousers; brown shoes, black patent-leather shoes, white patent-leather shoes, and sneakers.

How many different ensembles are possible today if he has already determined that he won't wear any stripes and he will wear everything that is white patent-leather?

- A. 288 B. 144 C. 15 D. 16

10. There are eight finalists in the Miss Sopchoppy contest. How many different outcomes are possible if one person will be selected First Runner-Up and another will be Miss Sopchoppy?

- A. 56 B. 64 C. 16 D. 256

11. Jethro has just purchased a hamster to be a companion for his gerbil, D.W. He is going to name the hamster by choosing a first name and second name from this list of names: Billy, Bobby, Bubba, Brett. The first name may or may not differ from the second name. How many different two-part names are possible?

- A. 8 B. 12 C. 16 D. 64

12. There are four Gators in a holding cell at the jail. They will be asked to arrange themselves from left to right in a police line-up. How many different line-ups are possible?

- A. 16 B. 8 C. 24 D. 256

13. Aristotle is going to register for classes next term. He has to choose one class from each of the following categories:

I. MGF1107, MAC1105, STA1014

II. REL2000, REL2121, REL2213

III. AFH1000, AMH1000, ASH3100, EUH2100

IV. LIT2020, LIT2081, LIT3043

How many different combinations of classes are possible if he has already decided that he will take LIT3043 and he won't take EUH2100 and he won't take REL2000?

- A. 108 B. 19 C. 24 D. 18

14. Harpo, Groucho, Chico, Zeppo, Gummo and Karl are the finalists in the fabulous Clearers' Publishinghouse sweepstakes. Two prizes will be awarded: the Grand Prize and the Even Grander Prize. The prizewinners will be randomly selected (and it is possible that one person may win both prizes). How many outcomes are possible?

- A. 720 B. 128 C. 30 D. 36

UNIT 3 MODULE 1

15. Euclid needs to create a password for his Internet account. To simplify things, he is going to form a four-letter password by choosing letters from this set: {A, D, E, M, R, S, T}. How many different passwords are possible if the second and fourth letters will be vowels and the other two letters won't be vowels (repeated letters are possible)?

- A. 40 B. 100 C. 256 D. 14

16. Socrates is going to buy a new pencil sharpener. He finds that the following six optional accessories are available: titanium bearings, fuel injection, CD rom, chromium alloy blades, air bags, HEPA filter shavings collector. How many different accessory combinations are possible?

- A. 64 B. 36 C. 720 D. 12

ANSWERS TO LINKED EXAMPLES

EXAMPLE 3.1.4 1. 180 2. 30

EXAMPLE 3.1.5 12

EXAMPLE 3.1.6 D

EXAMPLE 3.1.7 45,697,600

EXAMPLE 3.1.8 32 different ways to answer the 5 questions. The likelihood of getting all five questions correct by guessing is 1 out of 32.

EXAMPLE 3.1.9 4^{25} or roughly 1,126,000,000,000,000. The likelihood of getting all 25 questions right by guessing is 1 out of 1,126,000,000,000,000.

EXAMPLE 3.1.10 D

EXAMPLE 3.1.11 1. 500 2. 200

EXAMPLE 3.1.12 54

EXAMPLE 3.1.14 C

EXAMPLE 3.1.16 D

EXAMPLE 3.1.17 2912

EXAMPLE 3.1.18 64

EXAMPLE 3.1.19 D

EXAMPLE 3.1.20 620

ANSWERS TO HOMEWORK PROBLEMS

- | | | | | | |
|--------------|--------------|--------------|--------------|--------------|--------------|
| 1. B | 2. C | 3. D | 4. A | 5. D | 6. C |
| 7. D | 8. A | 9. B | 10. A | 11. C | 12. C |
| 13. D | 14. D | 15. B | 16. A | | |