

# INTRODUCTION

## How To Use This Text

As you should know, *Hacking Mathematics* is available on the World Wide Web at <http://www.math.fsu.edu/~wooland/hm2ed/hm2ed.html>

The various modules of this printed version of the text all have a similar structure. First, the course material is presented via examples and associated defining facts. Some of the examples show detailed, worked-out solutions, while most do not. The detailed solutions to these examples can be found in the web-page version of the text.

Toward the end of each module in the print version is a collection of practice exercises for that module. If you wish to do well on quizzes and tests, you should try to do all of these practice exercises, and you should ask your instructor for help with the exercises that you cannot solve correctly.

After the practice exercises you will find "Answers to Linked Examples." This shows the correct answers (but not detailed solutions) to the examples from the beginning part of the module.

Finally, you will find the "Answers to Practice Exercises" for that module.

It is important that you understand that the web-page version of the text is much more content-rich than this print version. For one thing, it contains links to detailed solutions for ALL of the examples (but not for the practice exercises). If you are studying on your own, you should find that the web-page version of the text is more helpful than the print version (except that only the print version has the practice exercises).

Finally, throughout the text there are references to "the companion web site." The companion web site for this text is <http://www.math.fsu.edu/~wooland/>

# PART 1 MODULE 1

## Set Mathematics

### Sets, Elements, Subsets

Any collection of objects can be considered to be a set.

We can define particular sets by listing the objects in each set. It is conventional to use set braces when doing so.

Examples of sets:

$\{a, b, c, d, e, f\}$

$\{4, 9, 2, 7\}$

$\{\text{Larry, Moe, Curly}\}$

$\{\text{cat, dog}\}$

$\{1, 2, 3, 4, 5, 6, 7, \dots\}$

$\{\text{the people in this room}\}$

#### NAMES FOR SETS

We will typically use upper case letters for the names of sets.

Let

$A = \{a, b, c, d, e, f\}$

$B = \{4, 9, 2, 7\}$

$C = \{\text{Larry, Moe, Curly}\}$

$D = \{\text{cat, dog}\}$

$N = \{1, 2, 3, 4, 5, 6, 7, \dots\}$

$F = \{\text{the people in this room}\}$

Note: the set  $\{1, 2, 3, 4, 5, 6, 7, \dots\}$  is very familiar to mathematicians and students of mathematics. It is usually called the set of **natural numbers**.

### ELEMENTS OR MEMBERS OF SETS

The contents of a set are called its **elements** or **members**.

We say "9 is an element of B"

"cat is an element of D"

Notation:

$$9 \in B$$

$$\text{cat} \in D$$

Likewise:

6 is not an element of B

$$6 \notin B$$

dog is not an element of A

$$\text{dog} \notin A$$

The **CARDINALITY** of a set is the number of elements in the set.

In general the cardinality of a set S is denoted  $n(S)$ .

For example, the cardinality of set B is 4.

Notation:

$$n(B) = 4$$

$$\text{Likewise } n(A) = 6$$

$$n(D) = 2$$

$$n(F) \text{ is } \underline{\hspace{2cm}}$$

$$n(N) \text{ is infinity?}$$

### SET EQUALITY

In any area of mathematics, a key goal is to determine when two objects that may look different from one another are actually the same object. In set mathematics, equality is defined in terms of the contents of sets:

*Two sets are said to be **equal** if they contain exactly the same elements.*

Examples

$$\{a, b, c\} = \{b, c, a\}$$

$$\{4, 2, 7, 9\} = B$$

On the other hand:

$\{1, 2, 3, 4\} \neq B$

$\{\text{Larry, Moe}\} \neq C$

**A formal definition of set equality** is this:

*Sets  $S$  and  $T$  are equal if every element of  $S$  is an element of  $T$  and every element of  $T$  is an element of  $S$ .*

### **SET-BUILDER NOTATION**

is a formalistic way of describing the elements of a set without listing them all. It is useful in some cases where a less formal description might be ambiguous. Here is an example of set-builder notation:

#### **EXAMPLE 1.1.1**

$F = \{x \mid x \text{ is a person in this room}\}$

We read this as follows:

" $F$  is the set of all  $x$  such that  $x$  is a person in this room"

Notice that in set-builder notation the construction " $\{x \mid \dots\}$ " is associated with the phrase "the set of all  $x$  such that..." This phrase is then followed by a description or characteristic that categorizes all the elements of the set. It has nothing to do with whether the symbol " $x$ " is one of the elements of the set.

More examples of set-builder notation:

$N = \{x \mid x \text{ is a positive integer}\}$

" $N$  is the set of all  $x$  such that  $x$  is a positive integer"

New set:

$E = \{x \mid x \text{ is a positive even number}\}$

List the elements of  $E$ .

**SUBSETS**

Let  $A = \{a, b, c, d, e, f\}$

$B = \{4, 9, 2, 7\}$

$C = \{\text{Larry, Moe, Curly}\}$

$D = \{\text{cat, dog}\}$

$E = \{2, 4, 6, 8, 10, \dots\}$

$N = \{1, 2, 3, 4, 5, 6, 7, \dots\}$

$F = \{x \mid x \text{ is a person in this room}\}$

$G = \{b, c, f\}$

Notice, for instance, that every element of  $G$  is also an element of  $A$ .

In a case like this we say "**G is a subset of A**"

Notation:

$G \subseteq A$

Likewise,

$B \subseteq N$

**A formal definition** of the word **subset** is this:

*For sets  $S$  and  $T$ ,  $S$  is a **subset** of  $T$  if every element of  $S$  is also an element of  $T$ .*

*This means that  $S$  is contained within  $T$ .*

Example

$\{\text{cat}\} \subseteq D$

$A \subseteq \{a, b, c, d, e, f, g, h\}$

$F \subseteq \{x \mid x \text{ is a person in this building}\}$

Formally:  $S$  is **not** a subset of  $T$  if there is at least one element of  $S$  that is not an element of  $T$ .

## PART 1 MODULE 1

### EXAMPLE 1.1.2

Referring to the sets listed earlier, determine whether each statement is true or false.

1.  $7 \in B$

2.  $n(A) = a$

3.  $B \subseteq N$

4.  $\text{dog} \subseteq D$

5.  $\{7\} \in B$

6.  $\{6, 8, 10\} \subseteq E$

7.  $\{6, 8, 10\} \subseteq \{6, 8, 10\}$

Exercise #7 above illustrates a general fact about subsets:

***Every set is a subset of itself.***

### PROPER SUBSETS

If  $S$  is a subset of  $T$  but is not equal to  $T$ , we say that  $S$  is a **proper subset** of  $T$ .

Notation:  $S \subset T$

#### Examples:

$\{2, 9\} \subset B$

$\{1, 3, 5, 7, 9, \dots\} \subset N$

$\{\text{dog, cat}\} \not\subset D$

The last example illustrates this basic fact:

***No set is a proper subset of itself.***

### COMPLEMENTS, THE UNIVERSAL SET, AND THE EMPTY SET

Consider the set  $S = \{\text{Moe, Larry}\}$

Suppose we want to complete the following exercise:

***List all of the elements that are NOT in  $S$ .***

Are we supposed to list...

...the names of all film performers except Moe and Larry?

...the names of all stooges except Moe and Larry?

...every object on Earth except Moe and Larry?

Unless we establish some boundaries on the scope of this exercise, we cannot make sense of it.

To establish a frame of reference for a set problem, we can define a **universal set** ( $U$ ) for the problem:

*For any set problem or discussion, a **universal set** ( $U$ ) is a "larger" set that contains all of the elements that may be of interest in the discussion; in particular, the universal set at least contains all of the elements of all of the sets in the discussion.*

Returning to the example at hand:

Let  $U = \{\text{Moe, Larry, Curly, Shemp, Curly Joe}\}$

Then to list the elements that are not in  $S$ , we list *those elements of  $U$  that are not in  $S$* :

$\{\text{Curly, Shemp, Curly Joe}\}$

This set is called the **complement of  $S$** , and is denoted  $S'$ .

### THE EMPTY SET

There is a unique set that contains no elements. It is called the **empty set** or the **null set** or the **void set**.

The empty set is usually denoted with one of these symbols:  $\{ \}$  or  $\emptyset$

**EXAMPLE 1.1.3**

Let  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$X = \{2, 4, 6, 8, 10\}$

$W = \{x|x \text{ is an odd number}\}$

$Y = \{3\}$

True or false:

1.  $X' = W$

2.  $Y \subseteq W$

3.  $Y \subset W$

4.  $X \subseteq U$

5.  $\{\} \subseteq X$

Exercises #4 and #5 above illustrate the following:

**FACT**

*In any set problem, every set is a subset of U, and  $\{\}$  is a subset of every set.*

**WORLD WIDE WEB NOTE**

For practice on problems involving sets, elements, subsets and the empty set, visit the companion website and try THE SETS APPEALER.

**THE NUMBER OF SUBSETS IN A FINITE SET**

General observation: It makes sense to assume that the more elements a set has, the more subsets it will have.

For finite sets, there is a strict relationship between the cardinality of a set and the number of subsets . We can discover this relationship by filling in the following table:

S	n(S)	Subsets of S	Number of subsets
$\{\}$			
$\{a\}$			
$\{a,b\}$			
$\{a, b, c\}$			
$\{a, b, c, d\}$			
$\{a, b, c, d, e\}$			



The completed table above suggests the following general fact:

**THE NUMBER OF SUBSETS IN A FINITE SET**

If  $n(S) = k$ , then the number of subsets in  $S$  is  $2^k$ .

**EXAMPLE 1.1.4.A**

Let  $A = \{a, b, c, d, e, f\}$  How many subsets does  $A$  have?

**SOLUTION**

Since  $n(A) = 6$ ,  $A$  has  $2^6$  subsets. That is,  $A$  has 64 subsets ( $2^6 = 64$ ).

**EXAMPLE 1.1.4.B**

Referring to set  $A$  above, how many proper subsets does  $A$  have?

**SOLUTION**

We already know that  $A$  has 64 subsets. Of those 64 subsets, the only one that is not a proper subset is set  $A$  itself (recall that while every set is a subset of itself, no set is a proper subset of itself). Thus,  $A$  has 63 proper subsets. This example suggests the following general fact:

*For any finite set, the number of proper subsets is one less than the number of subsets.*

**EXAMPLE 1.1.5**

Let  $U = \{1, 2, 3, 4, 5, 6, 7, \dots\}$

Let  $S = \{x \mid x \text{ is less than } 10\}$

1. How many subsets does  $S$  have?
2. How many proper subsets does  $S$  have?

**WORLD WIDE WEB NOTE**

For exercises involving the number of subsets and the number of proper subsets, visit the companion website and try THE SUBSETIZER.

**PRACTICE EXERCISES**

**1 – 17: Decide whether each statement is true or false.**

Let  $U = \{ 1, 2, 3, 4, 5, 6, 7 \}$   $S = \{1, 2, 4, 7\}$

$T = \{1, 2, 4, 5, 6, 7\}$   $V = \{4, 5, 6\}$

- |                                                      |                            |                            |                    |                    |
|------------------------------------------------------|----------------------------|----------------------------|--------------------|--------------------|
| 1. $S \subset T$                                     | 2. $n(S) = 7$              | 3. $4 \notin V$            | 4. $T \subseteq V$ | 5. $S \subseteq T$ |
| 6. $\{2, 4, 6\} = \{x x \text{ is an even number}\}$ | 7. $T \subseteq \{ \}$     | 8. $7 \in T$               |                    |                    |
| 9. T has 36 subsets                                  | 10. V has 7 proper subsets | 11. $\{ \} \subseteq V$    |                    |                    |
| 12. $V = \{x x > 3\}$                                | 13. $V = \{1, 2, 3\}$      | 14. $\{7, 4, 2, 1\} = S$   |                    |                    |
| 15. $\{1\} \subset T$                                | 16. $\{1\} \subseteq T$    | 17. $V \subseteq \{5, 6\}$ |                    |                    |

**ANSWERS TO LINKED EXAMPLES**

**EXAMPLE 1.1.2**

- |         |          |         |          |          |         |
|---------|----------|---------|----------|----------|---------|
| 1. True | 2. False | 3. True | 4. False | 5. False | 6. True |
| 7. True |          |         |          |          |         |

**EXAMPLE 1.1.3**

- |         |         |         |         |         |
|---------|---------|---------|---------|---------|
| 1. True | 2. True | 3. True | 4. True | 5. True |
|---------|---------|---------|---------|---------|

**EXAMPLE 1.1.5**

- |        |        |
|--------|--------|
| 1. 512 | 2. 511 |
|--------|--------|

**ANSWERS TO PRACTICE EXERCISES**

- |          |           |           |          |          |
|----------|-----------|-----------|----------|----------|
| 1. True  | 2. False  | 3. False  | 4. False | 5. True  |
| 6. True  | 7. False  | 8. True   | 9. False | 10. True |
| 11. True | 12. False | 13. False | 14. True | 15. True |
| 16. True | 17. False |           |          |          |