Prove: For integers $a$, $b$ and positive integer $m$, 
$a \equiv b \pmod{m}$ if and only if $a \mod m = b \mod m$

Proof:

(Note: because this theorem is a biconditional, we must prove it in “both directions.”)

First, assume $a \equiv b \pmod{m}$.

Then $m \mid (a - b)$, so there is $k \in \mathbb{Z}$ such that $a - b = mk$.

Let $a \mod m = r$.

Then, according to the division algorithm, there is $q \in \mathbb{Z}$ such that $a = mq + r$, $0 \leq r < m$.

Using $a = mq + r$ to replace $a$ in $a - b = mk$ we get

$mq + r - b = mk$

So

$mq - mk + r = b$ 

$m(q - k) + r = b$

This shows that $r$ is the remainder when $b$ is divided by $m$, so $b \mod m = r$ ($= a \mod m$).

We have proven that if $a \equiv b \pmod{m}$ then $a \mod m = b \mod m$. 
Conversely, assume \( a \mod m = b \mod m \).

Let \( r = a \mod m = b \mod m \).

Then, according to the division algorithm, there are \( q_1, q_2 \in \mathbb{Z} \) such that

\[
\begin{align*}
  a &= mq_1 + r, \\
  b &= mq_2 + r, \quad 0 \leq r < m.
\end{align*}
\]

Then

\[
\begin{align*}
  a - b &= mq_1 + r - (mq_2 + r) \\
        &= mq_1 + r - mq_2 - r \\
        &= mq_1 - mq_2 \\
        &= m(q_1 - q_2)
\end{align*}
\]

This shows that \( m \mid (a - b) \), so \( a \equiv b \pmod{m} \).

We have proven that

if \( a \mod m = b \mod m \) then \( a \equiv b \pmod{m} \).

This concludes the proof.