Prove: For any finite set S, if |S| = n, then S has $2^n$ subsets.

Proof by induction.

Let P(n) be the predicate “A set with cardinality n has $2^n$ subsets.

Basis step:
P(0) is true, because the set with cardinality 0 (the empty set) has 1 subset (itself) and $2^0 = 1$.

Inductive step

Prove P(k) → P(k+1)

That is, prove that if a set with k elements has $2^k$ subsets, then a set with k+1 elements has $2^{k+1}$ subsets.

Proof
Assume that for an arbitrary k, any set with cardinality k has $2^k$ subsets.

Let $T$ be a set such that |T| = k+1.

Enumerate the elements of $T$: $T = \{e_1, e_2, ..., e_k, e_{k+1}\}$.

Let $S = \{e_1, e_2, ..., e_k\}$

Then |S| = k, so S has $2^k$ subsets, according to the inductive hypothesis.

Note that $T = S \cup \{e_{k+1}\}$, so every subset of S is also a subset of T.

Any subset of T either contains the element $e_{k+1}$, or it doesn't contain $e_{k+1}$.

If a subset of T doesn't contain $e_{k+1}$, then it is also a subset of S, and there are $2^k$ of those subsets.

On the other hand, if a subset of T does contain the element $e_{k+1}$, then that subset is formed by including $e_{k+1}$ in one of the $2^k$ subsets of S, so T has $2^k$ subsets containing $e_{k+1}$.

We have shown that T has $2^k$ subsets containing $e_{k+1}$, and another $2^k$ subsets not containing $e_{k+1}$, so the total number of subsets of T is $2^k + 2^k = (2)2^k = 2^{k+1}$.