Course Notes Chapter 6 – Graph Theory

Digraphs

We are already familiar with simple directed graphs (usually called digraphs) from our study of relations.

More formally and generally, a digraph can be defined as follows, using the concepts of set mathematics:

Digraph - formal definition

A simple directed graph $G = (V, E)$ consists of a nonempty set representing vertices, $V$, and a set of ordered pairs of elements of $V$ representing directed edges, $E$. 
Example

Referring to the digraph $G$ shown above, $G = (V, E)$ where

$V = \{a, b, c, d, e\}$

$E = \{(a, d), (a, b), (a, c), (b, a), (c, c), (d, a), (e, c)\}$

The main idea here is that we can use a set of ordered pairs of vertices to represent the set of edges of a digraph.

We are already familiar, from our study of relations, with the connection matrix representation of a digraph.

A connection matrix is also called an adjacency matrix, because vertices connected by an edge are said to be adjacent.
**Simple graphs**

The most basic style of graph is a *simple undirected graph*, usually just called a *simple graph*.

Informally, a *simple graph* is a graph that has no loops, no arrows, and no multiple edges joining vertices.
1. Define a simple graph as a set.

The set of edges of simple graph can be defined as a set of unordered pairs.

Simple graph (formal definition)

A simple graph $G = (V, E)$ consists of a nonempty set representing vertices, $V$, and a set of unordered pairs of elements of $V$ representing edges, $E$.

The simple graph $G$ on the previous page can be represented as the following set:

$G = (V, E)$ where

$V = \{a, b, c, d, e\}$

$E = \{\{a, b\}, \{a, d\}, \{b, d\}, \{d, e\}\}$

The main idea here is that we can use unordered pairs (each of which is itself a two-element set) to represent the edges of a simple graph.

Note that the undirected edge $\{a, b\}$ is the same as the edge $\{b, a\}$, for example, just as, in set mathematics, the set $\{a, b\}$ is the same as the set $\{b, a\}$. 
**Adjacency matrix for a simple graph**

A simple graph can be represented with an $n \times n$ matrix, in a way similar to the way a digraph can be represented by a matrix, with one important difference:

*If there is a 1 in row $x$ column $y$, there must also be a 1 in row $y$ column $x$.*

Both ones represent the same edge: the edge $\{x, y\}$.

Because of this, the connection matrix for an undirected graph is always symmetric with respect to the main diagonal.

Also, because there are no loops on a simple graph, every entry along the main diagonal is 0.
Graphs with multiple (or parallel) edges

While most of our focus is on simple graphs and digraphs, there are also graphs with multiple edges.

Multiple or parallel edges are:

Two or more undirected edges connecting the same pair of vertices on an undirected graph;

or

Two or more directed edges connecting the same pair of vertices in the same direction on a directed graph.
More complicated graph styles

Undirected

A multigraph is like a simple graph, except that it has multiple edges (but no loops).
A *pseudograph* is like a simple graph, except that it has loops (and may have multiple edges).
Directed
A directed multigraph is like a digraph, except that it has multiple edges.
On the adjacency matrix for a graph with multiple edges, some entries will be greater than 1:

The integer $n$ in row $x$ column $y$ on the adjacency matrix for an undirected graph means that there are $n$ undirected edges connecting vertex $x$ with vertex $y$.

Make the adjacency matrix for this multigraph.

Matrix:

\[
\begin{bmatrix}
0 & 1 & 3 \\
1 & 0 & 2 \\
3 & 2 & 0 \\
\end{bmatrix}
\]
The integer \( n \) in row \( x \) column \( y \) on the adjacency matrix for a directed graph means that there are \( n \) directed edges initiating at vertex \( x \) and terminating at vertex \( y \).

Make the adjacency matrix for this **directed multigraph**:

```
Matrix:

0 0 3
2 0 0
1 0 1
```