Chapter 1: Heat equation

Chapter 2: Separation of variables

Chapter 3: Fourier series

Chapter 4: Wave equation: vibrating strings and membranes

Chapter 5: Sturm-Liouville Eigenvalue Problems

Chapter 7: Higher Dimensional PDEs

Chapter 8: Nonhomogeneous Problems
What are PDEs?
Why do we care about PDEs?
What are the basic issues (existence, uniqueness and continuous dependence)?
Derivations of the 1D heat equation.

- thermal energy density $e(x, t)$: heat energy per unit volume
- temperature $u(x, t)$
- specific heat / heat capacity $c$: heat energy needed to raise the temperature of a unit mass substance by one unit
- $e = c\rho u$
- heat flux $\phi(x, t)$: rate of thermal energy flowing to the right per unit mass per unit time
- heat source $Q(x, t)$: heat energy generated per unit volume per unit time.
- conservation of heat energy: changing rate of heat energy = heat flux in - heat flux out + heat energy generated inside per unit time
- Fourier’s law for heat conduction

$$\phi(x, t) = -K_0 \frac{\partial u}{\partial x}(x, t)$$

thermal conductivity $K_0$

- 1D heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + q$$

thermal diffusivity $\kappa = \frac{K_0}{c\rho}$

(read the part on diffusion of chemical pollutant, Fick’s law of diffusion.)
Aug. 28th lecture (sect. 1.3, 1.4 and 1.5.)

- 1d heat equation: \( \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + q \)

- initial condition for the 1d heat equation

- boundary condition for the 1d heat equation
  - prescribed temperature \( u(0, t) = u_B(t) \) (Dirichlet),
  - prescribed flux \(-K_0(0) \frac{\partial u}{\partial x}(0, t) = \phi(t)\) (Neumann),
  - Newton’s law of cooling \(-K_0(0) \frac{\partial u}{\partial x}(0, t) = -H[u(0, t) - u_b(t)]\) (Robin) (heat transfer coefficient \( H \))

- Equilibrium temperature in 1D with prescribed temperature or insulated boundaries, convergence to equilibrium solution

- Derivation of the heat equation in 2 and 3 D via conservation of heat energy

\[
\frac{\partial u}{\partial t} = \kappa \Delta u + q
\]

(notation difference: \( \Delta = \nabla \cdot \nabla = \nabla^2 \))

- steady state
  - Poisson’s equation
    \[ \Delta u = f \]
  - Laplace equation \((f = 0)\) (potential equation)
    \[ \Delta u = 0 \]
separation of variables for 1d homogeneous heat equation

\[
\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(L, t) = 0, \ u(x, 0) = f(x)
\]

- simple special solution via separation of variables: \( u(x, t) = G(t)\phi(x) \)
- associated time-dependent equation
- associated boundary value problem, and eigenvalue and eigenfunctions problem
  \(-\phi''(x) = \lambda \phi(x), \ \phi(0) = \phi(L) = 0\)
- special solution \( u_n(x, t) = \sin \frac{n\pi x}{L} e^{-\kappa(n\pi/L)^2 t} \)
- strategy for general solution (sect. 2.2)
  - linear operator, linear (homogeneous) equation
  - principle of superposition for linear homogeneous equation
- construction of (formal) general solution via principle of superposition

\[
u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\kappa(n\pi/L)^2 t}, \quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \ dx
\]

(and convergence to steady state)

- Fourier sine series \( f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \), completeness (form a basis), orthogonality of the sines

- \( L^2 \) inner product: \( (f, g) \overset{\text{def}}{=} \int_0^L f(x)g(x), \ L^2 \) length: \( \|f\|_{L^2} = \sqrt{(f, f)}. \) (not in the textbook)
- Example: initial temperature = 100° constant: \( B_n = 0 \) if \( n \) is even, and \( B_n = \frac{400}{n\pi} \) if \( n \) is odd. (as a HW)
- Issues: completeness (convergence)? basis? differentiability? (postponed)
1d homogeneous heat equation \( \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, u(x, 0) = f(x) \) with different b.c.

- with insulated ends \( \frac{\partial u}{\partial x} \bigg|_{x=0,L} = 0 \)
  - separation of variables and the associated eigenvalue-eigenfunction problem
  - **Fourier cosine series** \( f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \)
  - formal solution (and convergence to steady state)

\[
    u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-\kappa(n\pi/L)^2 t},
\]

\[
    A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dx, \quad A_n = \frac{2}{L} \int_0^L u(x, 0) \cos \frac{n\pi x}{L} \, dx
\]

- (completeness) and orthogonality of the Fourier cosine series
  - \( f(x) = x, A_n = \frac{2L}{(n\pi)^2} (\cos n\pi - 1), n = 1, 2, \ldots, A_0 = \frac{L}{2} \)

- in a thin circular ring and periodic boundary condition
  - \( (u(-L, t) = u(L, t), \frac{\partial u}{\partial x} (-L, t) = = \frac{\partial u}{\partial x} (L, t)) \)
  - separation of variables and the associated eigenvalue-eigenfunction problem
  - **Fourier series** \( f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \)
  - formal solution (and convergence to steady state)

\[
    u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} e^{-\kappa(n\pi/L)^2 t} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-\kappa(n\pi/L)^2 t},
\]

\[
    a_0 = \frac{1}{2L} \int_{-L}^L u(x, 0) \, dx, \quad a_n = \frac{1}{L} \int_{-L}^L u(x, 0) \cos \frac{n\pi x}{L} \, dx
\]

\[
    b_n = \frac{2}{L} \int_0^L u(x, 0) \sin \frac{n\pi x}{L} \, dx
\]

- (completeness) and orthogonality of the Fourier series
  - \( f(x) = 0, x < \frac{L}{2}; f(x) = 1, x > \frac{L}{2}, \) \( a_0 = \frac{1}{4}, a_n = \frac{1}{n\pi} (\sin n\pi - \sin \frac{n\pi}{2}), b_n = \frac{1}{n\pi} (\frac{n\pi}{2} - \cos n\pi) \)
Summary of various eigenvalue-eigenfunction problems $-\varphi'' = \lambda \varphi + \text{b.c.}$

<table>
<thead>
<tr>
<th>BC</th>
<th>Dirichlet (prescribed ...</th>
<th>Neumann (insulated ends)</th>
<th>periodic (1d ring)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E'val</td>
<td>$(n\pi/L)^2, n = 1, 2, 3, \cdots$</td>
<td>$(n\pi/L)^2, n = 0, 1, 2, 3, \cdots$</td>
<td>$(n\pi/L)^2, n = 0, 1, 2, 3, \cdots$</td>
</tr>
<tr>
<td>E'fnct</td>
<td>$\sin \frac{n\pi x}{L}$</td>
<td>$\cos \frac{n\pi x}{L}$</td>
<td>$\sin \frac{n\pi x}{L}, \cos \frac{n\pi x}{L}$</td>
</tr>
<tr>
<td>series</td>
<td>$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$</td>
<td>$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$</td>
<td>$f(x) = a_0$ + $\sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$ + $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$</td>
</tr>
<tr>
<td>coeff</td>
<td>$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}$</td>
<td>$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}$</td>
<td>$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)$, $a_n = \frac{1}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}$, $b_n = \frac{1}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}$</td>
</tr>
</tbody>
</table>

- Questions: convergence? completeness? differentiability?
- What about mixed boundary conditions, say Dirichlet at one end, and Neumann at the other end? (optional)
- Alternative view via eigenfunction expansion (assume orthogonality)

\[
\text{finite dim'} | \text{infinite dim'}
\]

| eq | $\frac{d\tilde{u}}{dt} + A\tilde{u} = 0, \tilde{u}\big|_{t=0} = \tilde{u}_0 \in R^n$ | $\frac{\partial u}{\partial t} + \kappa Au = 0, u\big|_{t=0} = f$ |
| e'val& e'vec (e'fnct) | $\lambda_j, \tilde{v}_j, (A\tilde{v}_j = \lambda_j \tilde{v}_j), j = 1, \cdots, n$ | $\lambda_j, \varphi_j, j = 1, 2, \cdots$ |
| special soln | $\tilde{v}_j e^{-\lambda_j t}$ | $\varphi_j(x)e^{-\kappa \lambda_j t}$ |
| general soln | $\sum_{j=1}^{n} c_j \tilde{v}_j e^{-\lambda_j t}$ | $\sum_{j=1}^{\infty} c_j \varphi_j(x)e^{-\kappa \lambda_j t}$ |

\[
u(x, t) = \sum_{j=1}^{\infty} C_j(t)\varphi_j(x), u(x, 0) = f(x) \Rightarrow C_j(t) = c_j e^{-\kappa \lambda_j t}, c_j = \frac{(f, \varphi_j)}{\langle \varphi_j, \varphi_j \rangle}
\]
Sept. 11th lecture (sect. 2.5.1)
Laplace equation $\Delta u = 0$ inside a rectangle

- **Boundary conditions:**

  $BC1(\text{left}) : u(0, y) = g_1(y);$
  $BC2(\text{right}) : u(L, y) = g_2(y);$
  $BC3(\text{bottom}) : u(x, 0) = f_1(x);$
  $BC4(\text{top}) : u(x, H) = f_2(x)$

- **Divide and conquer** $u = u_1 + u_2 + u_3 + u_4$ with $u_j$ takes care of $BC_j$ (different from the textbook!)

- **Example:** $BC1$, i.e., $u_1$ via separation of variables $u_1(x, y) = h(x)\phi(y)$

  $$u_1(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi(x - L)}{H}$$

  $$A_n = \frac{2}{H \sinh n\pi(-L)/H} \int_0^H g_1(y) \sin \frac{n\pi y}{H} \, dy$$

  Special case of $g_1(y) = 1$, $A_n = \frac{8}{nH\pi \sinh n\pi(-L)/H}$ for odd $n$. $A_n = 0$ otherwise.

- **Alternative view via eigenfunction expansion** (covered on Sept. 9th)

  $$u_1(x, y) = \sum_{n=1}^{\infty} C_n(x) \sin \frac{n\pi y}{H} \Rightarrow C_n''(x) - \left(\frac{n\pi}{H}\right)^2 C_n(x) = 0 \quad \Rightarrow \quad C_n(x) = A_n \sinh \frac{n\pi(x - L)}{H}$$
Sept. 11th lecture continued (sect. 2.5.2)
Laplace’s equation $\Delta u = 0$ on a circular disk with b.c. $u |_{r=a} = f(\theta)$
- Laplace operator in polar coordinates
  \[ \Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \]
- separation of variable $u(r, \theta) = \phi(\theta)G(r)$
- associated eigenvalue-eigenfunction problem: $-\phi'' = \lambda \phi$ plus periodic b.c.
- radial function equation $r \frac{d}{dr} (rG'(r)) - n^2 G(r) = 0$ (Euler type)
- solution formula
  \[
  u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta
  \]

1. $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta \quad (= u(\vec{0}))$
2. $A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$
3. $B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$

- Example: $f(\theta) = \sin \theta$. $A_n = 0$, $B_n = 0$ if $n \neq 1$, $B_1 = \frac{1}{a}$, $u(x, y) = \frac{y}{a}$.
- Alternative view via eigenfunction expansion
  \[
  u(r, \theta) = G_0(r) + \sum_{n=1}^{\infty} (G_{ns}(r) \sin n\theta + G_{nc}(r) \cos n\theta)
  \]
Sept. 16th lecture (section 2.5.4)

- Qualitative property of harmonic functions (solution to Laplace equation)
  - Statement and understanding of mean value theorem: $u(\vec{0}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a, \theta) \, d\theta$
  - Maximum principle for harmonic functions
  - Uniqueness of solution to Poisson equation and continuous dependence on the boundary data (in $L^\infty$ norm)
- Solvability (compatibility) condition for Poisson equation in the Neumann BC case.
- Energy method for well-posedness of 1d heat equation (postponed)
- $L^2$ length and Fourier coefficients: $\| \sum B_n \sin \frac{n\pi x}{L} \|_{L^2} = \sqrt{\frac{L}{2} \sum B_n^2}$ (postponed)
My mathematical pedigree according to the Mathematics Genealogy Project:
My mathematical lineage

Joseph-Louis Lagrange (student of Leonhard Euler) joint with Pierre-Simon Laplace (student of Jean Le Rond d’Alembert)
⇒
Siméon Denis Poisson (Ph.D. École Polytechnique 1800) + Jean Baptiste Joseph Fourier + ...
⇒
Michel Chasles (Ph.D. École Polytechnique 1814) + Johann Peter Gustav Lejeune Dirichlet (joint with Fourier) + Joseph Liouville
⇒
Jean Gaston Darboux (Ph.D. École Normale Supérieure Paris 1866) + another 1
⇒
Félix Edouard Justin Émile Borel (Ph.D. École Normale Supérieure Paris, 1893) + ...
⇒
Georges Valiron (Ph.D. Université de Paris, 1914) + ...
⇒
Laurent Schwartz (Ph.D. Université Louis Pasteur - Strasbourg I, 1943, Fields medalist 1950) + ...
⇒
Jacques-Louis Lions (Ph.D. Université Henri Poincaré Nancy 1, 1954) + Alexander Grothendieck + ...
⇒
Roger M. Témam (Ph.D. Université de Paris, 1967) + ...
⇒
Yours truly (Ph.D. Indiana University - Bloomington, 1996) + ...
Sept. 18th lecture, (Sect. 3.1, 3.2, Fourier series and convergence statement)

- Fourier series (defines a function if it converges)

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = f(x)
\]

- Fourier coefficients (for a function \( f \))

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad b_n = \frac{1}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\]

\[
f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (=?)
\]

- piecewise smooth function, jump discontinuity, periodic extension

- Point-wise convergence Theorem: assume piece-wise smoothness; converges to the average at a jump discontinuity \( \frac{1}{2} [f(x+) + f(x-)] \)

- Sketching Fourier series: 1. periodic ext.; 2. mark an "x" at the average of the two values at any jump discontinuity.

- Example: \( f(x) = 1 \) if \( x > L/2 \); \( f(x) = 0 \) otherwise.

\[
a_0 = \frac{1}{4}, \quad a_n = \frac{1}{n\pi} \left( \sin n\pi - \sin \frac{n\pi}{2} \right), \quad b_n = \frac{1}{n\pi} \left( \cos \frac{n\pi}{2} - \cos n\pi \right)
\]

- \( L^2 \) theory (not covered in the textbook)

  - condition for convergence: \( \sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty \)

  - condition on the function for convergence: \( \|f\|_{L^2} < \infty \)

  - relationship between a function and its Fourier series: \( f \overset{L^2}{=} \) its Fourier series,

\[
\|f\|_{L^2}^2 \overset{def}{=} \int_{-L}^{L} f^2(x) = L(2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2))
\]
Fourier sine series $f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$, $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}$

Odd function, odd extension, odd part of a function and the associated Fourier sine series

Point-wise convergence Theorem: Assume piece-wise smoothness with jump discontinuity, the Fourier sine series converges to $\frac{1}{2} [f(x+) + f(x-)]$ for the extended (odd, then periodic) function.

Sketching Fourier sine series: 1. odd extension; 2. periodic ext.; 3. mark an "x" at the average of the two values at any jump discontinuity.

Example with Gibbs phenomenon: $f(x) = 100$, $B_n = \frac{400}{n\pi}$ for odd $n$, 0 otherwise. Gibbs phenomenon (9% overshoot for $f(x) = 100$ with matlab code, or low modes, 51 modes, figures on p.103)

More examples $f(x) = x$, $B_n = \frac{2L}{n\pi} (-1)^{n+1}$; $f(x) = \cos \frac{\pi x}{L}$, $B_n = \frac{4n}{\pi(n^2-1)}$ for even $n$.

$L^2$ theory (not covered in the textbook)

- condition for convergence: $\sum_{n=1}^{\infty} B_n^2 < \infty$
- condition on the function for convergence: $\|f\|_{L^2} < \infty$

- relationship between a function and its Fourier series: $f \equiv^2$ its Fourier sine series, $\|f\|_{L^2}^2 = \frac{L}{2} \sum_{n=1}^{\infty} B_n^2$
**Gibbs phenomenon**

Figure: Fourier sine series for \( f(x) = 100 \) with 1, 3, 5, and 7 terms

Figure: Fourier sine series for \( f(x) = 100 \) with 51 terms
Fourier cosine series

\[ f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad A_0 = \frac{1}{L} \int_0^L f(x) \, dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \]

Point-wise convergence Theorem: piece-wise smooth function with jump discontinuity only, F-cosine series converges to \( \frac{1}{2} [f(x+) + f(x-)] \) of the extended function (even then periodic)

Sketching Fourier cosine series: 1. even extension; 2. periodic ext.; 3. mark an "x" at the average of the two values at any jump discontinuity.

Examples: \( f(x) = x; \) \( f(x) = -\frac{L}{2} \sin \frac{\pi x}{L} \) if \( x < 0, f(x) = x \) for \( 0 < x < \frac{L}{2} \), \( f(x) = L - x \) if \( x > \frac{L}{2} \).

\( L^2 \) theory (not covered in the textbook)

- condition for convergence: \( \sum_{n=0}^{\infty} A_n^2 < \infty \)
- condition on the function for convergence: \( \| f \|_{L^2} < \infty \)

relationship between a function and its Fourier series: \( f \Leftrightarrow \) its Fourier cosine series,
\[ \| f \|_{L^2}^2 = \frac{L}{2} \left( 2A_0^2 + \sum_{n=1}^{\infty} A_n^2 \right) \]

Even part of a function and the associated Fourier cosine series

Fourier series = Fourier sine series for the odd part + Fourier cosine series for the even part.
Sept. 23 continued (3.3.5, 3.6, cont F-series and term-by-term diff)

- Continuous Fourier series
  - general F-series: piecewise smooth + continuous + periodicity
  - F-cosine: piecewise smooth + continuous
  - F-sine: piecewise smooth + continuous + \( f(0) = f(L) = 0 \)

- term-by-term differentiation of Fourier series
  - Counter-example to term-by-term differentiation
    \( x = 2L \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{L} \), but
    \( 1 \neq 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi x}{L} \)
  - point-wise sense
    - general F-series: continuity of the F-series + piecewise smoothness of \( f' \).
      Alternative form on \( f \): continuity of \( f \) + periodicity + piecewise smoothness
    - F-cosine: piecewise smoothness of \( f' \) + continuity of the F-cosine series.
      Alternative form on \( f \): piecewise smoothness of \( f' \) + continuity of \( f \).
    - F-sine: piecewise smoothness of \( f' \) + continuity of F-sine series.
      Alternative: piecewise smoothness of \( f' \) + continuity of \( f \) but no \( f(0) = f(L) = 0 \) (Postponed)

\[
f'(x) \sim \frac{1}{L} [f(L) - f(0)] + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} B_n + \frac{2}{L} ((-1)^n f(L) - f(0)) \right] \cos \frac{n\pi x}{L}
\]

- Example: \( f(x) = x \). F-cosine: \( x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos \frac{n\pi x}{L} \).
- Example \( f(x) = x \). F-sine series (w/o \( f(0) = f(L) = 0 \)): \( x \sim \sum_{n=1}^{\infty} \frac{L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L} \).
  \[
  \frac{df}{dx} \sim 1
  \]

\( L^2 \) sense differentiability (NOT in the textbook)

\[
\sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) < \infty
\]

- Application to 1d homogeneous heat equation: solution is infinitely many times differentiable w.r.t. \( x \) for any positive time.
Sept. 25 and 30 lecture, (3.4, diff w.r.t parameter, $L^2$ theory)

- $L^2$ sense differentiability (NOT in the textbook)
  \[
  \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) < \infty
  \]

- Differentiation with respect to parameters, $L^2$ theory only (not in the textbook)
  \[
  \sum_{n=1}^{\infty} \left( |a_n'(t)|^2 + |b_n'(t)|^2 \right) < \infty
  \]

- Application of the $L^2$ theory to 1d homogeneous heat equation with zero temperatures at the ends (not in the textbook)
  - Solution is infinitely many times differentiable w.r.t. $x$ and $t$ for any positive time with $f \in L^2$
    \[
    u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\kappa(n\pi/L)^2 t},
    B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx
    \]
  
  - Solution does solve the heat equation
  - Solution is unique (via energy method)
  - Solution continuously depends on the initial data (via energy method)

- Other boundary conditions and comparisons
Sept. 30 and Oct. 2nd lecture, (3.5, 3.6, integration, complex form)

- term-by-term integration of Fourier series: always okay and always converges (s3.5)

\[
\int_{-L}^{x} f(t) dt = a_0(x + L) + \sum_{n=1}^{\infty} \left[ \frac{a_n}{n\pi/L} \sin \frac{n\pi x}{L} + \frac{b_n}{n\pi/L} (\cos n\pi - \cos \frac{n\pi x}{L}) \right]
\]

- Example \( f(x) = 1 \sim \frac{4}{\pi} \left( \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots \right) \)

\[
x \sim \frac{4L}{\pi^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) - \frac{4L}{\pi^2} \left( \cos \frac{\pi x}{L} + \frac{1}{3^2} \cos \frac{3\pi x}{L} + \frac{1}{5^2} \cos \frac{5\pi x}{L} + \cdots \right)
\]

\[
\frac{4L}{\pi^2} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) = \frac{1}{L} \int_{0}^{L} x dx = \frac{L}{2}
\]

- complex form of Fourier series and complex orthogonality (s3.6)

\[
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L}, (c_0 = a_0, c_n = \frac{a_n + ib_n}{2}, n > 0)
\]

- complex orthogonality: \( \int_{-L}^{L} e^{-in\pi x/L} e^{-im\pi x/L} dx = 0 \) if \( n \neq m \)

- convergence in the \( L^2 \) sense: \( \sum |c_n|^2 < \infty \)

- once-differentiability in the \( L^2 \) sense: \( \sum n^2 |c_n|^2 < \infty \)

- Example: \( f(x) = |x| \).

\[
c_0 = \frac{L}{2}, c_{2k-1} = -\frac{2L}{(k-1)^2 \pi^2}, c_{2k} = 0, k = 1, 2, \ldots, c_{-n} = c_n (b_n = 0)
\]
Oct. 7 lecture, (1D wave equation: derivation and solution 4.1, 4.2, 4.3)

- derivation of the equation (1D wave) of a vertically vibrating string via Newton’s law
  - vertical displacement: $u(x, t)$
  - mass density: $\rho_0(x)$
  - tensile force: $T(x, t) = \text{tension for perfectly flexible strings}$
  - angle between the horizontal and the string: $\theta(x, t)$, $(\frac{\partial u}{\partial x} = \frac{dy}{dx} = \tan \theta \approx \sin \theta)$
  - Newton’s law in the vertical direction
    $$\rho_0(x)\delta x \frac{\partial^2 u}{\partial t^2} = T(x + \delta x, t) \sin \theta(x + \delta x, t) - T(x, t) \sin \theta(x, t) + \rho_0(x)\delta x Q(x, t)$$
  - wave equation:
    $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c = \sqrt{\frac{T_0}{\rho_0(x)}} \quad \text{(wave speed)}$$

- boundary conditions
  - fixed ends: $u(0, t) = u(L, t) = 0$ (more generally: prescribed ends) (Dirichlet)
  - elastic $T_0 \frac{\partial u}{\partial x}(0, t) = k(u(0, t) - l). \ (y(t) = u(0, t): \text{attached to a spring with spring constant } k, \ m \frac{\partial^2 y}{\partial t^2} = -k(y(t) - l) + T_0 \frac{\partial u}{\partial x}(0, t) + f_{\text{ext}}. \text{ For } 1 \gg m, f_{\text{ext}} = 0, \text{ leads to elastic}$
  - free: $T_0 \frac{\partial u}{\partial x}(0, t) = 0$ (Neumann).

- solution of the 1d wave equation via separation of variables
  $$u(x, t) = \sum_{n=1}^{\infty} \left( A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi c t}{L} + B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi c t}{L} \right)$$
  $$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}, \quad u(x, 0) = f(x), \quad B_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}, \quad \frac{\partial u}{\partial t}(x, 0) = g(x)$$

- solution to inhomogeneous 1D wave equation with fixed ends via eigenfunction expansion (not in the textbook).
Oct. 9th lecture: (tentative, 4.3, 4.4, 4.5)

- normal modes (harmonics, overtones): \( \sin \frac{n\pi x}{L} (A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}) \)
- natural frequencies: \( n\pi c/L, n = 1, 2, \cdots \)
- standing waves: motion of the normal modes

- nodes: zeros of the harmonics
- traveling waves (right and left with speed \( c \)):

\[
\sin \frac{n\pi x}{L} \sin \frac{n\pi ct}{L} = \frac{1}{2} \cos \frac{n\pi}{L} (x - ct) - \frac{1}{2} \cos \frac{n\pi}{L} (x + ct)
\]

- differentiability of the solution to the wave equation;
- Derivation of 2D wave equation (vibrating membrane)
  - vertical displacement \( z = u(x, y, t) \)
  - mass density \( \rho_0(x, y) \)
  - tensile force \( \vec{F}_T = T_0 \vec{t} \times \vec{n} \)
  - Newton’s law and Stokes theorem \( \iint \nabla \times \vec{B} \cdot \vec{n} dA = \oint \vec{B} \cdot \vec{t} ds \)

\[
\iint \rho_0 \frac{\partial^2 u}{\partial t^2} dA = \oint T_0 \vec{t} \times \vec{n} \cdot \vec{k} ds = \oint T_0 (\vec{n} \times \vec{k}) \cdot \vec{t} ds
\]

\[
\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \nabla \times (\vec{n} \times \vec{k})
\]

\[
\vec{n} \approx -u_x \vec{i} - u_y \vec{j} + \vec{k}
\]

- Conservation of total energy, uniqueness of solutions and continuous dependence on data (not in the textbook)
Oct. 14 lecture (tentative, 4.6, Reflection and refraction of electromagnetic/light and acoustic/sound wave)

- plane traveling wave \( u = A e^{i(\vec{k} \cdot \vec{x} - \omega t)} \), wave vector \( \vec{k} \), wave number \( k \overset{\text{def}}{=} |\vec{k}| \), wave length (\( \overset{\text{def}}{=} \frac{2\pi}{k} \)), wave front (\( \vec{k} \cdot \vec{x} - \omega t = \text{constant} \)), phase velocity (\( \overset{\text{def}}{=} \frac{\omega}{k} = \pm c \))

- dispersive relationship for wave equation: \( \omega^2 = c^2 k^2 \)

- incident wave \( e^{i(\vec{k}_I \cdot \vec{x} - \omega_+(k_I) t)} \), reflected wave \( Re^{i(\vec{k}_R \cdot \vec{x} - \omega_+(k_R) t)} \), refracted (transmitted) wave \( Te^{i(\vec{k}_T \cdot \vec{x} - \omega_-(k_T) t)} \). Continuity at \( z = 0 \):

\[
e^{i(\vec{k}_I \cdot \vec{x} - \omega_+(k_I) t)} + Re^{i(\vec{k}_R \cdot \vec{x} - \omega_+(k_R) t)} = Te^{i(\vec{k}_T \cdot \vec{x} - \omega_-(k_T) t)}
\]

- Consequences
  - \( \omega_+(k_I) = \omega_+(k_R) = \omega_-(k_T) \Rightarrow k_I = k_R \Rightarrow k_{I,z} = -k_{R,z} \Rightarrow \theta_I = \theta_R \)
  - Snell's law of refraction (from horizontal frequency consideration \( k_I \sin \theta_I = k_T \sin \theta_T \))

\[
\frac{\sin \theta_T}{\sin \theta_I} = \frac{k_I}{k_T} = \frac{c_-}{c_+}
\]

- evanescent wave for large incident angle and \( c_+ < c_- \)
October 23 lecture (examples and statements of S-L problems, 5.2, 5.3)

- Examples of Sturm-Liouville problem
  - Heat flow in a non-uniform rod with a source term depending on solution $Q = \alpha u$
    \[
    \frac{d}{dx} \left( K_0 \frac{d\phi}{dx} \right) + \alpha \phi + \lambda c \rho \phi = 0
    \]
  - Circularly symmetric heat flow
    \[
    \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) + \lambda r \phi = 0
    \]

- Regular (continuous plus $p > 0, \sigma > 0$ on $[a, b]$) Sturm-Liouville eigenvalue problem
  \[
  \frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x) \phi + \lambda \sigma(x) \phi = 0, \ a < x < b,
  \beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0, \ \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0
  \]

- Statement of theorems for regular S-L problem
  1. All eigenvalues are real.
  2. There exist an infinite number of eigenvalues: $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$
  3. Corresponding eigenfunction $\phi_n(x)$ (unique up to an arbitrary multiplicative constant) with $n - 1$ zeros in $(a, b)$.
  4. \{\phi_n, n = 1, 2, \cdots \} form a "basis" or a "complete" set in the sense of the generalized Fourier series
    \[
    f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)
    \]
    for any piecewise smooth function (convergence to average at each jump discontinuity).
  5. Eigenfunctions are mutually orthogonal relative to the weight function $\sigma(x)$, i.e.,
    \[
    \int_a^b \phi_n(x) \phi_m(x) \sigma(x) dx = 0, \text{ if } \lambda_n \neq \lambda_m
    \]
  6. Rayleigh quotient $\lambda = \left( -p \phi \frac{d\phi}{dx} \right)_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2] dx / \int_a^b \phi^2 \sigma dx$
Oct. 28 lecture, (Examples of Sturm-Liouville problem and applications, 5.3, 5.4)

- Example of Sturm-Liouville problem: heat conduction in a uniform rod without heat source with zero temperature at the end. 5.3.3
- Application of Sturm-Liouville problem to heat conduction in a non-uniform rod (variable conductivity and density, formal manipulation). 5.4.

Eigenfunction expansion!
Oct. 28 and 30 lecture, (Self-adjoint operators and S-L eigenvalue problems, 5.5.)

Goal: motivate and prove a few properties for regular \((p > 0, \sigma > 0)\) S-L problems:

\[
\frac{d}{dx} (p(x) \frac{d\phi}{dx}) + q(x)\phi + \lambda \sigma(x)\phi = 0, \quad a < x < b,
\]

\[
\beta_1 \phi(a) + \beta_2 \frac{d\phi}{dx}(a) = 0, \quad \beta_3 \phi(b) + \beta_4 \frac{d\phi}{dx}(b) = 0
\]

- \(L(u) = \frac{d}{dx}(p(x) \frac{du}{dx}) + q(x)u\) is self-adjoint, i.e., \(\int_{a}^{b}[uL(v) - vL(u)]dx = 0\) (Lagrange identity \(uL(v) - vL(u) = \frac{d}{dx}[p(x)(u \frac{dv}{dx} - v \frac{du}{dx})]\))
- eigenfunctions corresponding to different eigenvalues are mutually orthogonal with weight \(\sigma(x)\)
- all eigenvalues are real
- unique eigenfunction
- non-unique eigenfunction in the periodic case
- Gram-Schmidt orthogonalization
Oct. 30 lecture, (S-L eigenvalue problem with Robin b.c., 5.8.)

\[-\frac{d^2}{dx^2}\phi = \lambda \phi, \quad \phi(0) = 0, \quad h\phi(L) + \frac{d\phi}{dx}(L) = 0\]

▶ positive eigenvalues: \(\tan \sqrt{\lambda}L = -\frac{\sqrt{\lambda}}{h}\)
  ▶ physical case \(h > 0\)

▶ nonphysical case \(h < 0\) (two major subcases: \(-1/hL > 1\) and \(0 < -1/hL < 1\), and a minor subcase: \(-1/hL = 1\))

▶ zero eigenvalue: \(\phi(x) = x\) if \(hL = -1\), not an eigenvalue otherwise.

▶ negative eigenvalues: \(\tanh \sqrt{s}L = -\frac{\sqrt{s}}{h}\), \(s = -\lambda\)

▶ special case \(h = 0\): \(\lambda_n = (n - \frac{1}{2})^2 \pi^2 / L^2\)
Oct. 30 lecture, (S-L eigenvalue problem with Robin b.c. summary and app, 5.8.)

<table>
<thead>
<tr>
<th></th>
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<th>$\lambda = 0$</th>
<th>$\lambda &lt; 0$</th>
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<td>$\sin \sqrt{\lambda} x$</td>
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<tr>
<td>physical</td>
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<tr>
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<td>$hL &lt; -1$</td>
<td>$\sin \sqrt{\lambda} x$</td>
<td>$\sinh \sqrt{s_1} x$</td>
</tr>
</tbody>
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Application: heat flow with a nonphysical boundary condition

\[
\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = -hu(L, t), \quad u(x, 0) = f(x), \quad hL < -1
\]

\[
\phi_n(x) = \begin{cases} 
\sinh \sqrt{s_1} x, & n = 1, \\
\sin \sqrt{\lambda_n} x, & n > 1
\end{cases}
\]

\[
u(x, t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n \kappa t}, \quad f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)
\]

\[
a_n = \frac{\int_0^L f(x) \phi_n(x) dx}{\int_0^L \phi_n^2 dx} = \begin{cases} 
\frac{\int_0^L f(x) \sinh \sqrt{s_1} x dx}{\int_0^L \sinh^2 \sqrt{s_1} x dx}, & n = 1 \\
\frac{\int_0^L f(x) \sin \sqrt{\lambda_n} x dx}{\int_0^L \sin^2 \sqrt{\lambda_n} x dx}, & n \geq 2
\end{cases}
\]
"best approximation" of a function $f$ with finitely many terms from the basis $\{\phi_n, n = 1, 2, \cdots \}$

- Mean-square deviation

\[
E_M(\alpha_1, \cdots, \alpha_M) = \int_a^b [f(x) - \sum_{n=1}^M \alpha_n \phi_n(x)]^2 \sigma(x) \, dx.
\]

- Minimizing mean-square deviation leads to

\[
\alpha_n = a_n = \frac{\int_a^b f(x) \phi_n(x) \sigma(x) \, dx}{\int_a^b \phi_n^2(x) \sigma(x) \, dx}
\]

- Approximation error

\[
E_M = \int_a^b f^2 \sigma \, dx - \sum_{n=1}^M \alpha_n^2 \int_a^b \phi_n^2 \sigma \, dx
\]

- Bessel's inequality

\[
E_M(a_1, \cdots, a_n) \geq 0
\]

- Parseval's identity (equality)(generalization of Pythagorean)

\[
\int_a^b f^2 \sigma \, dx = \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma \, dx
\]
Nov. 6th lecture, (separation of time variable, 7.1, 7.2)

- separation of time variable (vibrating membrane and heat conduction in arbitrary region)

- eigenvalue-eigenfunction problem for the Laplace operator

\[-\Delta \phi = \lambda \phi\]
\[a\phi + b\nabla \phi \cdot \vec{n} = 0\]

Special case of a rectangular domain with homogeneous Dirichlet boundary condition

- vibrating rectangular membrane with fixed boundary and
  \[u(x, y, 0) = \alpha(x, y), \frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y)\]

  - separation of variables: \(u(x, y, t) = \phi(x, y)h(t) = f(x)g(y)h(t)\)

    \[\phi_{nm}(x, y) = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\]

- general solution (and 2D F-series)

  \[u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{nm} \cos c\sqrt{\lambda_{nm}}t + B_{nm} \sin c\sqrt{\lambda_{nm}}t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H},\]

  \[\alpha(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad A_{nm} = \frac{2}{L} \int_{0}^{L} \frac{2}{H} \int_{0}^{H} \alpha(x, y) \sin \frac{m\pi y}{H} dy \sin \frac{n\pi x}{L} dx,\]

  \[c\sqrt{\lambda_{nm}}B_{nm} = \frac{4}{LH} \int_{0}^{L} \int_{0}^{H} \beta(x, y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} dy dx,\]
Nov. 13th lecture (statement of Theorems for eigenvalue problem for Laplace operator, 7.4)

- statement of the eigenvalue problem for the Laplace operator

\[-\Delta \phi = \lambda \phi\]
\[a \phi + b \nabla \phi \cdot \vec{n} = 0\]

1. All the eigenvalues are real.
2. There exists an infinite number of eigenvalues. There is a smallest one, but no largest one.
3. Dimension of eigenspace may be more than one.
4. The eigenfunctions \(\phi\) form a "complete" set (one can have a "basis" that consists of eigenfunctions only), i.e, we have generalized F-series

\[f(x, y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x, y)\]

5. Eigenfunctions corresponding to different eigenvalues are mutually orthogonal.
6. Rayleigh quotient

\[\lambda = -\oint \phi \nabla \phi \cdot \vec{n} + \iint_R |\nabla \phi|^2 \, dx \, dy \quad \frac{\iint_R \phi^2 \, dx \, dy}{\iint_R \phi^2 \, dx \, dy}\]

- example of Laplace operator on a rectangle with homogeneous Dirichlet boundary condition

\[\phi_{nm}(x, y) = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad n, m = 1, 2, \ldots\]

- application to vibrating rectangular membrane

\[u = \sum_{n,m} a_{nm}(t) \phi_{nm}(x, y)\]
Goal: prove or motivate the results for eigenvalue problems for the Laplace operator

\[-\Delta \phi = \lambda \phi, (x, y) \in R, \quad a\phi + b\nabla \phi \cdot \vec{n} = 0, (x, y) \in \partial R\]

- Green's formula (2nd identity)

\[
\iint_{R} (u\Delta v - v\Delta u) \, dx \, dy = \oint (u \nabla v - v \nabla u) \cdot \vec{n} \, ds
\]

- \(L = -\Delta\) plus b.c is self-adjoint.
- All eigenvalues are real.
- Eigenfunctions corresponding to different eigenvalues are mutually orthogonal.
- Rayleigh quotient.

Appendix: Gram-Schmidt orthogonalization

- \(\phi_1, \ldots, \phi_n\): independent eigenfunctions corresponding to the same eigenvalue
- G-S procedure product: \(\psi_1, \ldots, \psi_n\) mutually orthogonal eigenfunctions for the same eigenvalue

Description:

1. \(\psi_1 = \phi_1\)

2. For \(j = 2, \ldots, n\),

\[
\psi_j = \phi_j - \sum_{i=1}^{j-1} \frac{\iint_{R} \phi_j \psi_i \, dx \, dy}{\iint_{R} \psi_i^2 \, dx \, dy} \psi_i
\]

- Example: \(\phi_1 = 1, \phi_2 = x, \phi_3 = x^2\) on \([0, 1]\)
Motivation via separation of variable for eigenvalue problem ($-\Delta \phi = \lambda \phi$) on a disk: $\phi(r, \theta) = f(r)g(\theta)$ (a singular S-L problem for $f$)

$$\frac{d^2 g}{d\theta^2} = -\mu g, \quad \frac{1}{r} \frac{d}{dr} \left( r \frac{df}{dr} \right) + (\lambda - \frac{\mu}{r^2}) f = 0, (|f(0)| < \infty, f(a) = 0)$$

Bessel’s DE of order $m$

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0 \quad (r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + (\lambda r^2 - m^2) f = 0, z = \sqrt{\lambda} r)$$

Bessel function 1st and 2nd kind of order $m$ (asymptotic properties near $z = 0$)

$$J_m(z) \sim \begin{cases} \frac{1}{2^m m!} z^m & m = 0 \\ \frac{1}{2^m m!} z^m & m > 0 \end{cases}, \quad Y_m(z) \sim \begin{cases} \frac{2}{\pi} \ln z & m = 0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m > 0. \end{cases}$$

Large $z$ asymptotics: $f(z) \sim \sin z, \cos z \Rightarrow$ infinitely many zeros (denoted $z_{mn}$ for the 1st kind) (more precise result in 7.8.1)

Eigenvalue-eigenfunction problems (1D S-L):

$$f_{mn}(r) = J_m \left( \frac{z_{mn} r}{a} \right), (f = J_m(\sqrt{\lambda} r), J_m(\sqrt{\lambda} a) = 0, \lambda_{mn} = \frac{z_{mn}^2}{a^2})$$

$$\alpha(r) = \sum_{n=1}^{\infty} a_n J_m(\sqrt{\lambda_{mn}} r), a_n = \frac{\int_0^a \alpha(r) J_m(\sqrt{\lambda_{mn}} r) r dr}{\int_0^2 J_m^2(\sqrt{\lambda_{mn}} r) r dr},$$

Eigenvalue-eigenfunction for the Laplace operator on a disk (with Dirichlet bc)

$$\phi_{mn}(r, \theta) = J_m(\sqrt{\lambda_{mn}} r) \left\{ \begin{array}{ll} \cos m\theta & \text{for } m > 0 \\ \sin m\theta & \text{for } m = 0 \end{array} \right\}$$
Nov. 18th lecture (Bessel’s function applications to vibrating circular membrane, 7.7)

- Application to vibrating circular membrane \( \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \): \( u(r, \theta, t) = f(r)g(\theta)h(t) \)
  
  (or just eigenfunction expansion \( u(r, \theta, t) = \sum \alpha \lambda \phi \lambda (r, \theta) h_\lambda (t) \))

\[
J_m(\sqrt{\lambda_{mn}r}) \left\{ \begin{array}{c}
\cos m\theta \\
\sin m\theta
\end{array} \right\} \left\{ \begin{array}{c}
\cos c\sqrt{\lambda_{mn}t} \\
\sin c\sqrt{\lambda_{mn}t}
\end{array} \right\}
\]

- Special case of \( \frac{\partial u}{\partial t} (r, \theta, 0) = \beta (r, \theta) = 0 \)

\[
u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}r}) \cos m\theta \cos c\sqrt{\lambda_{mn}t} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}r}) \sin m\theta \cos c\sqrt{\lambda_{mn}t}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} \phi_{mnc}(r, \theta) \cos c\sqrt{\lambda_{mn}t} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \phi_{mns}(r, \theta) \cos c\sqrt{\lambda_{mn}t}
\]

\[
\alpha (r, \theta) = \sum_{\lambda} A_{\lambda} \phi_{\lambda} (r, \theta), \quad A_{\lambda} = \frac{\iint \alpha (r, \theta) \phi_{\lambda} (r, \theta) \, dA}{\iint \phi_{\lambda}^2 (r, \theta) \, dA}
\]

- circular symmetric case \( u(r, 0) = \alpha (r), \frac{\partial u}{\partial t} (r, 0) = \beta (r) \) \((\lambda_n = \lambda_{0n})\)

\[
u(r, t) = \sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_nr}) \cos c\sqrt{\lambda_nt} + \sum_{n=1}^{\infty} b_n J_0(\sqrt{\lambda_nr}) \sin c\sqrt{\lambda_nt}
\]

\[
a_n = \frac{\int_0^a \alpha (r) J_0(\sqrt{\lambda_nr}) r \, dr}{\int_0^a J_0^2(\sqrt{\lambda_nr}) r \, dr}, \quad c\sqrt{\lambda_n} b_n = \frac{\int_0^a \beta (r) J_0(\sqrt{\lambda_nr}) r \, dr}{\int_0^a J_0^2(\sqrt{\lambda_nr}) r \, dr}
\]
Eigenvalue-eigenfunction problem for the Laplace operator, Sturm-Liouville problems, special functions

- higher dimensional PDEs \((u_t = \kappa \Delta u, u_{tt} = c^2 \Delta u) \) plus bc and ic, Laplace eq.
- reduction idea and relationship to the eigenvalue-eigenfunction problem for the L-operator
  - separation of variables (temp and spat) \(u = h(t) \phi(\vec{x}) \) \(\Rightarrow\) eigenvalue-eigenfunction prob for L-operator \(\Rightarrow\) \(\lambda, \phi_\lambda(\vec{x}) \) \(\Rightarrow\) (superposition) \(u = \sum_\lambda \alpha_\lambda h(\lambda) \phi_\lambda(\vec{x}) \) (why?)
  - freeze time variable, expand in terms of eigenfunctions (why?) \(u = \sum_\lambda u_\lambda(t) \phi_\lambda(\vec{x}) \)
    \((u_\lambda(t) = \alpha_\lambda h(\lambda(t)) \) \(\Rightarrow\) (for the heat eq) \(\frac{d}{dt} u_\lambda(t) = -\lambda \kappa u_\lambda(t) \) (why?) \(\Rightarrow\)
    \(u_\lambda(t) = e^{-\lambda \kappa t} u_\lambda(0) \Rightarrow u = \sum_\lambda u_\lambda(0) e^{-\lambda \kappa t} \phi_\lambda(\vec{x}) \) \(\Rightarrow\) (IC)
    \(\alpha(\vec{x}) = u(\vec{x}, 0) = \sum_\lambda u_\lambda(0) \phi_\lambda(\vec{x}) \Rightarrow\) (orthogonality) \(u_\lambda(0) = \frac{\int \alpha(\vec{x}) \phi_\lambda(\vec{x}) d\vec{x}}{\int \phi_\lambda^2(\vec{x}) d\vec{x}}\)
- eigenvalue-eigenfunction problem for the L-operator
  - 1D case: (standard) regular S-L problem with trigonometric functions as eigenfunctions \(\Rightarrow\) solution = Fourier series
  - 2D case (further reduction needed). 1. case rectangular domain (separation of variable or using 1D F-series): 2D eigenfunctions are double Fourier series. 2. case circular disk (separation of variables in polar coord or using 1D F-series in \(\theta\)) \(\Rightarrow\) Bessel's equation (a singular S-L problem) in the radial direction and Bessel functions \(\Rightarrow\) (bc) Bessel's function of the 1st kind \(J_m\) and eigenvalues \(\lambda_{mn} = z_{mn}^2 / a^2 \) \(\Rightarrow\) eigenfunctions
    \(J_m(\sqrt{\lambda_{mn}} r) \cos m\theta, J_m(\sqrt{\lambda_{mn}} r) \sin m\theta.\)
  - 3D case (additional reduction needed) 1. case of a rectangular box \(\Rightarrow\) triple Fourier series. 2. case of a sphere \(\Rightarrow\) Legendre functions and polynomials. \(P_n^m\) and eigenfunctions
    \(\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda} \rho) \left\{ \begin{array}{l} \cos m\theta \\ \sin m\theta \end{array} \right\} \quad P_n^m(\cos \phi), \lambda_{n+1/2,l} = \frac{z_{n+1/2,l}^2}{a^2} \)
Nov. 25th lecture (Legendre polynomials and functions, 7.10)

- Laplace operator eigenvalue-eigenfunction problem \((-\Delta w = \lambda w)\) in a sphere in spherical coordinates \((x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi)\)

\[
\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial w}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial w}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 w}{\partial \theta^2} + \lambda w = 0
\]

- Use separation of variables \((w = \tilde{w}(\rho, \phi)q(\theta) = f(\rho)q(\theta)g(\phi))\) and \(2\pi\) periodicity in \(\theta\)

\[
\frac{d}{d\rho} \left( \rho^2 \frac{df}{d\rho} \right) + (\lambda \rho^2 - \mu)f = 0, \quad \frac{d}{d\phi} \left( \sin \phi \frac{dg}{d\phi} \right) + (\mu \sin \phi - \frac{m^2}{\sin \phi})g = 0
\]

- Associated Legendre functions and Legendre polynomials

\((z = \cos \phi, dz = -\sin \phi d\phi)\)

\[
\frac{d}{dz} \left[ (1 - z^2) \frac{dg}{dz} \right] + (\mu - \frac{m^2}{1 - z^2})g = 0, \quad (\text{singular S – L prob})
\]

\(\mu = n(n + 1)\) in order to ensure boundedness at \(z = \pm 1\) (mysterious) \(\Rightarrow\)
\(g(z) = P^n_m(z)\)

- \(m = 0\): Legendre polynomials:

\(P_0(z) = 1, P_1(z) = z, P_2(z) = \frac{1}{2}(3z^2 - 1), P_n(z) = P^n_0(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n}(z^2 - 1)^n\)

- \(m > 0\): associated Legendre functions (spherical harmonics)

\[
P^n_m(z) = (z^2 - 1)^{m/2} \frac{d^m}{dz^m} P_n(z), \quad \int_{-1}^{1} [P^n_m(x)]^2 dx = (n + \frac{1}{2} - 1) (n + m) \frac{(n + m)!}{(n - m)!}
\]
Nov. 25th lecture (eigenvalue and eigenfunctions for L-operator on sphere, 7.10)

- The radial S-L problem

\[
\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda \rho^2 - n(n+1))f = 0, f(a) = 0, |f(0)| < \infty
\]

- Relationship to Bessel's function \(Z_p(x)\) (HW)

\[
f(\rho) = \rho^{-1/2}Z_{n+\frac{1}{2}}(\sqrt{\lambda}\rho) \quad \text{(spherical Bessel functions)}
\]

- boundedness \(\Rightarrow\)

\[
f(\rho) = \rho^{-1/2}J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho)
\]

- bc \(\Rightarrow\)

\[
\lambda = \lambda_{n+\frac{1}{2}}, J_{n+\frac{1}{2}}(\sqrt{\lambda_{n+\frac{1}{2}}}, a) = 0, l = 1, 2, \ldots
\]

- relation to trigonometric functions

\[
x^{-1/2}J_{n+\frac{1}{2}}(x) = x^n\left(-\frac{1}{x} \frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)
\]

- Eigenvalue and eigenfunction for the Laplace operator on the sphere with homogeneous bc

\[
\lambda = \lambda_{n+\frac{1}{2}}, n = 1, 2, \ldots, l = 1, 2, \ldots
\]

\[
w_{\lambda}(\rho, \phi, \theta) = \rho^{-1/2}J_{n+\frac{1}{2}}(\sqrt{\lambda}\rho) \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix} P_n^m(\cos \phi), \quad \text{form a basis}
\]

- surface harmonics

\[
\begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix} P_n^m(\cos \phi), n = 1, 2, \ldots, m = 0, 1, 2, \ldots
\]
Nov. 25th lecture (application to heat conduction in a sphere)

- The model

\[
\frac{\partial u}{\partial t} - \kappa \Delta u = 0, \quad u|_{\rho=a} = 0, \quad u(\rho, \phi, \theta, 0) = \alpha(\rho, \phi, \theta)
\]

- Use eigenfunction expansion (why?)

\[
u = \sum_{\lambda} u_\lambda(t) w_\lambda(\rho, \phi, \theta)
\]

\[
\lambda = \lambda_{n+\frac{1}{2}}, \quad n = 1, 2, \ldots, \quad l = 1, 2, \ldots, \quad (-\Delta w_\lambda = \lambda w_\lambda)
\]

\[
w_\lambda(\rho, \phi, \theta) = \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda} \rho) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} P_n^m(\cos \phi)
\]

- reduced equation

\[
u_\lambda' + \kappa \lambda u_\lambda = 0
\]

- initial condition

\[
\alpha = \sum_{\lambda} u_\lambda(0) w_\lambda(\rho, \phi, \theta) = \sum_{\lambda} \alpha_\lambda(t) w_\lambda(\rho, \phi, \theta)
\]

\[
\Rightarrow \quad u_\lambda(t) = \alpha_\lambda e^{-\kappa \lambda t}, \quad \alpha_\lambda = \frac{\int \int \int \alpha w_\lambda \, dV}{\int \int \int w_\lambda^2 \, dV}, \quad (dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta)
\]

- Final answer

\[
u(\rho, \phi, \theta, t) = \sum_{\lambda} \alpha_\lambda e^{-\kappa \lambda t} w_\lambda(\rho, \phi, \theta)
\]
Laplace equation in a cylinder

- Laplace operator in cylindrical coordinates
  \[ \Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \]

- Divide and conquer strategy \( u = u_1(\text{top bc}) + u_2(\text{bottom bc}) + u_3(\text{lateral bc}) \)
- Strategy for \( u_1 \) and \( u_2 \): eigenfunction expansion for the 2D Laplace operator in a disk
  \[ u_1 = \sum_{m,n} \left( A_{mn} J_m(\sqrt{\lambda_{mn}} r) \cos m\theta + B_{mn} J_m(\sqrt{\lambda_{mn}} r) \sin m\theta \right) h_{mn}(z), \quad h_{mn} = \sinh \sqrt{\lambda_{mn}} z \]

- Strategy for \( u_3 \):
  - Eigenfunction expansion for the 1D Laplace operator in the vertical direction first
    \( h_n(z) = \sin \frac{n\pi z}{H}, \lambda = (\frac{n\pi}{H})^2 \)
  - Separation of var in the horizontal variables \((r, \theta)\) and use \(2\pi\) periodicity in \(\theta\)
    \[ r \frac{d}{dr} \left( r \frac{df}{dr} \right) + \left( -\left( \frac{n\pi}{H} \right)^2 r^2 - m^2 \right) f = 0 \]
    \[ w^2 \frac{d^2f}{dw^2} + w \frac{df}{dw} + \left( -w^2 - m^2 \right) f = 0, \quad w = \frac{n\pi}{H} r \]
  - Modified Bessel’s functions: 1st kind (well-defined at \( w = 0 \)) \( I_m(w) \), 2nd kind (singular at the origin) \( K_m(w) \)
  - General solution (taking into consideration boundedness requirement)
    \[ u_3 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} E_{mn} I_m \left( \frac{n\pi}{H} r \right) \sin \frac{n\pi z}{H} \cos m\theta + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} I_m \left( \frac{n\pi}{H} r \right) \sin \frac{n\pi z}{H} \sin m\theta \]

- Properties of the modified Bessel functions (check Abramowitz and Stegun for more)
December 2nd lecture (non-homogeneous problem, 8.1, 8.2, 8.3)

How to deal with non-homogeneous problem? Reduction!

▶ Source term with homogeneous boundary condition: eigenfunction expansion (8.3). e.g. \( u_t = u_{xx} + \sin 3x \, e^{-t}, \, u(0, t) = u(\pi, t) = 0, \, u(x, 0) = f(x). \)

▶ Nonhomogeneous boundary condition: convert to a problem with homogeneous bc after subtracting a background profile \( b \) that satisfies the bc (sect.8.2). e.g. previous one with \( u(0, t) = 0, \, u(\pi, t) = 1 \Rightarrow b(x) = x/\pi. \)

▶ Eigenfunction expansion without converting to a problem with homogeneous bc: example of 1D heat equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \kappa u_{xx} + Q(x, t), \quad u(0, t) = A(t), \quad u(L, t) = B(t), \quad u(x, 0) = f(x). \\
\text{eigenfunction expansion} &\quad u = \sum_{n=1}^{\infty} b_n(t) \phi_n(x), \quad Q(x, t) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \\
\text{equation in terms of expansion} &\quad \sum_{n=1}^{\infty} \frac{db_n}{dt} \phi_n(x) = \kappa u_{xx} + \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \\
\text{equation in terms of the coefficients} &\quad \frac{db_n}{dt} + \kappa \lambda_n b_n = \kappa \frac{\kappa(n\pi/L)[A(t) - (-1)^n B(t)]}{\int_0^L \phi_n^2 \, dx} + q_n(t) \\
\text{potential downside with this approach?}
\end{align*}
\]

▶ Multi-dimensional case? Background profile could use Poisson equation. Source term could use eigenfunction expansion. Read section 8.4.

e.g. \( u_t = \kappa \Delta u + Q(\vec{x}, t), \, u|_{t=0} = f \) plus inhomogeneous bc. Let \( b \) be a background profile, say \( \Delta b = 0 \) with the same b.c. as \( u \) (how?). Consider \( v = u - b \Rightarrow v_t = \kappa \Delta v + Q(\vec{x}, t) - b_t \) plus homogeneous bc. Apply eigenfunction expansion method to find \( v \). \( u = v + b. \)