

A Note on Long Time Behavior of Solutions to the Boussinesq System at Large Prandtl Number

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ABSTRACT. We establish the eventual regularity of suitably defined weak solutions to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. The existence of a global attractor of the Boussinesq system at large Prandtl number is also presented.

1. Introduction

We consider here a convection problem, i.e., fluid motion induced by differential heating, of a layer of fluids bounded by two horizontal parallel plates a distance h apart in the Rayleigh-Bénard setting (Tritton 1988, Getling 1998). The governing equation, after the customary Boussinesq approximation and non-dimensionalization, is the following Boussinesq system

$$(1) \quad \frac{1}{Pr} \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla p = \Delta \mathbf{u} + Ra \mathbf{k} T, \quad \nabla \cdot \mathbf{u} = 0,$$

$$(2) \quad \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T,$$

$$(3) \quad \mathbf{u}|_{z=0,1} = 0,$$

$$(4) \quad T|_{z=0} = 1, \quad T|_{z=1} = 0,$$

$$(5) \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad T|_{t=0} = T_0,$$

where \mathbf{u} is the fluid velocity field, p is the pressure, T is the temperature field and \mathbf{k} is the unit upward vector. The fluids occupy the (non-dimensionalized) region

$$(6) \quad \Omega = [0, L_x] \times [0, L_y] \times [0, 1]$$

with periodicity in the horizontal directions assumed for simplicity. The parameters of the system are thus absorbed into the geometry of the domain plus two adimensional numbers: the Rayleigh number

$$(7) \quad Ra = \frac{g\alpha(T_2 - T_1)h^3}{\nu\kappa}$$

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measuring the ratio of overall buoyancy force to the damping coefficients; and the Prandtl number

$$(8) \quad Pr = \nu/\kappa$$

measuring the relative importance of kinematic viscosity over thermal diffusivity. Here ν and κ are the kinematic viscosity and thermal diffusive coefficient respectively, α is the thermal expansion coefficient of the fluid, g is the gravitational constant, h is the distance between the two plates confining the fluid, $T_2 - T_1$ is the temperature difference between the bottom and top plates, and we have taken the distance between the plates, h , as typical length scale, the thermal diffusive time as typical time scale, and the temperature is scaled so that the top plate is set to 0 while the bottom plate is set to 1 (see Tritton 1988 among others).

The Boussinesq system exhibits extremely rich phenomena from pure conduction at low Rayleigh number, to Bénard cells at first bifurcation, spatial-temporal patterns and chaos at intermediate Rayleigh number, all the way to convective turbulence at high Rayleigh number (see for instance Getling 1998, Tritton 1988, and the recent review by Bodenschatz, Pesch and Ahlers 2000, Siggia 1994). On the other hand, we have very limited mathematical knowledge on the system. Even the issue of the existence of regular enough solutions is unresolved. Indeed, the velocity equation is exactly the Navier-Stokes system (forced by a buoyancy term) whose regularity of solutions is one of the million dollar mathematical problems of the new millennium (www.claymath.org/Millennium_Prize_Problems/). For such a complex system, simplification is highly desirable. Simpler model can be obtained if we consider the regime of large Prandtl number. In this case we arrive at the so-called infinite Prandtl number model (see for instance Bodenschatz, Pesch and Ahlers 2000, Busse 1989, Chandrasekhar 1961, Constantin and Doering 1999, Grossmann and Lohse 2000, Tritton 1988, Wang 2003a, b among others) which is relevant for fluids such as silicone oil and the earth's mantle as well as many gases under high pressure. Mathematical analysis on the infinite Prandtl number model and related models on convections in porous media in terms of wellposedness and existence of global attractors can be found in Fabrie 1990, Fabrie and Nicolaenko 1996, Ly and Titi 1999, Marcel and Titi 2000 among others and the references therein. The mathematical justification of the infinite Prandtl number model on a short time interval can be found in Wang 2003b. On the other hand, no fluid has infinite Prandtl number. Thus it is of interest to investigate long time properties of the solutions of the Boussinesq system at large (but finite) Prandtl number. This is the purpose of this short note. In particular we will show that all suitably defined weak solutions to the Boussinesq system become regular after a transitional time period, and all these suitable weak solutions converge to a global attractor which is smooth, invariant, compact and connected. This is in sharp contrast to the 3D Navier-Stokes system (see for instance Constantin and Foias 1988, Ladyzhenskaya 1969, Majda and Bertozzi 2001, Temam 2000 among others). Such a global attractor is non-trivial as we shall discuss later. The existence of suitable weak solutions will also be briefly discussed.

2. Suitable weak solution and main results

To motivate our main result and the definition of suitable weak solution, we rewrite the Boussinesq system in terms of the viscous time scale ($\tau = Pr t$) and we

have

$$(9) \quad \frac{D\mathbf{v}}{D\tau} + \nabla p = \Delta\mathbf{v} + \frac{Ra}{Pr} \mathbf{k}T, \quad \nabla \cdot \mathbf{v} = 0,$$

$$(10) \quad \frac{\partial T}{\partial \tau} + \mathbf{v} \cdot \nabla T = \frac{1}{Pr} \Delta T,$$

$$(11) \quad T \Big|_{z=1} = 0, \quad T \Big|_{z=0} = 1,$$

$$(12) \quad \mathbf{v} \Big|_{z=0,1} = 0$$

where $\mathbf{u} = Pr\mathbf{v}$.

Heuristically we have, for the temperature field a maximum principle type of estimate which bounds the temperature field for all time, and then the velocity field is the Navier-Stokes system forced by forcing term proportional to $\frac{Ra}{Pr}$ which is small for large Prandtl number and fixed (but arbitrary) Rayleigh number. This inspires us to follow the work of J. Leray on the eventual regularity of solutions to the 3D decaying Navier-Stokes system and derive eventual regularity here. Moreover, there exists an absorbing ball in H^1 that attracts all bounded sets in the phase space. This further implies the existence of a global attractor.

Although the maximum principle for the temperature field is obvious for smooth solutions, it is not clear we have it for the Leray-Hopf type weak solutions. The remedy is then to build such a maximum type of inequality into the definition and this leads us to introduce the following definition of suitable weak solutions which is basically the Leray-Hopf definition plus maximum inequality.

DEFINITION 2.1. (\mathbf{u}, θ) is called a suitable weak solution to the Boussinesq equations on the time interval $[0, T^*]$ with given initial data (\mathbf{u}_0, θ_0) if the following hold

$$(13) \quad \mathbf{u} \in L^\infty(0, T^*; H) \cap L^2(0, T^*; V) \cap C_w([0, T^*]; H),$$

$$(14) \quad \mathbf{u}' \in L^{4/3}(0, T^*; V'), \mathbf{u}(0) = \mathbf{u}_0$$

$$(15) \quad \theta \in L^\infty(0, T^*; L^2(\Omega)) \cap L^2(0, T^*; H_0^1(\Omega)) \cap C_w([0, T^*]; L^2(\Omega)),$$

$$(16) \quad \theta' \in L^{4/3}(0, T^*; H^{-1}(\Omega)), \theta(0) = \theta_0$$

$$(17) \quad \frac{1}{Pr} \left(\frac{d}{dt} \int_\Omega \mathbf{u} \cdot \mathbf{v} + \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} \right) + \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} = Ra \int_\Omega \theta v_3, \quad \forall \mathbf{v} \in V$$

$$(18) \quad \frac{d}{dt} \int_\Omega \theta \eta + \int_\Omega \mathbf{u} \cdot \nabla \theta \eta + \int_\Omega \nabla \theta \cdot \nabla \eta = \int_\Omega u_3 \eta, \quad \forall \eta \in H_0^1(\Omega)$$

$$(19) \quad \frac{1}{Pr} |\mathbf{u}(t)|_{L^2}^2 + 2 \int_0^t |\nabla \mathbf{u}(s)|_{L^2}^2 ds \leq \frac{1}{Pr} |\mathbf{u}_0|_{L^2}^2 + 2Ra \int_0^t (\theta(s), u_3(s)) ds$$

$$(20) \quad |(T - 1)^+(t)|_{L^2}^2 + 2 \int_0^t |\nabla (T - 1)^+(s)|_{L^2}^2 ds \leq |(T_0 - 1)^+|_{L^2}^2,$$

$$(21) \quad |T^-(t)|_{L^2}^2 + 2 \int_0^t |\nabla T^-(s)|_{L^2}^2 ds \leq |(T_0)^-|_{L^2}^2.$$

Here we have used the perturbative variable (perturbation from the pure conduction state $(0, 1-z)$) $\theta = T - (1-z)$, and we used standard notations on function spaces used in the study of incompressible fluids (see for instance Constantin and

Foias 1988, Doering and Gibbon 1996, Temam 2000 among others). The existence of such suitable weak solutions will be briefly addressed later.

Our main results are

THEOREM 1. There exists a constant κ_0 depending on the domain only, such that for $\frac{Pr}{Ra} \geq \kappa_0$ and any initial data (\mathbf{u}_0, T_0) there exists a constant T^* so that all the suitable weak solutions with the given initial data to the Boussinesq equations become regular after T^* .

Moreover, we have

THEOREM 2. There exists a κ_0 depending on the domain only, such that for $\frac{Pr}{Ra} \geq \kappa_0$ the Boussinesq system possesses a global attractor which attracts all suitable weak solutions.

3. Eventual regularity

Here we sketch the proof of the eventual regularity of the solutions to the Boussinesq system at large Prandtl number.

We will provide a priori estimates that are suffice to deduce the eventual regularity of the solutions. The proof is similar to the proof of regularity of Leray-Hopf weak solutions to the 3D Navier-Stokes system under small data assumption. The interested reader can fill in the details (see for instance Constantin and Foias 1988, Ladyzhenskaya 1969, Temam 2000 among others).

It is easy to see, thanks to (20-21), the following is true

$$\begin{aligned} |T(t)|_{L^2} &\leq |(T(t) - 1)^+|_{L^2} + |T(t)^-|_{L^2} + |T(t) - (T(t) - 1)^+ + T(t)^-|_{L^2} \\ &\leq |(T_0 - 1)^+|_{L^2} + |T_0^-|_{L^2} + |\Omega| \\ (22) \quad &\leq \kappa_1(|T_0|). \end{aligned}$$

Utilizing this in the energy equation for the velocity field we have

$$\frac{1}{Pr} \frac{d}{dt} |\mathbf{u}(t)|_{L^2}^2 + |\nabla \mathbf{u}(t)|_{L^2}^2 \leq Ra^2 |T(t)|_{L^2}^2 \leq \kappa_1^2(|T_0|) Ra^2$$

which further implies, after utilizing Poincaré inequality and Gronwall inequality,

$$\begin{aligned} |\mathbf{u}(t)|_{L^2}^2 &\leq |\mathbf{u}_0|_{L^2}^2 e^{-Pr t} + Ra^2 \kappa_1^2(|T_0|) \\ (23) \quad &\leq |\mathbf{u}_0|_{L^2}^2 + Ra^2 \kappa_1^2(|T_0|), \end{aligned}$$

$$(24) \quad \frac{1}{T^*} \int_0^{T^*} |\nabla \mathbf{u}(t)|_{L^2}^2 dt \leq \frac{1}{T^*} \frac{1}{Pr} |\mathbf{u}_0|_{L^2}^2 + Ra^2 \kappa_1^2(|T_0|).$$

Next, we multiply the velocity equation by $A\mathbf{u}$ and integrate over the domain where A denotes the Stokes operator. We have

$$\begin{aligned} \frac{1}{2Pr} \frac{d}{dt} |\nabla \mathbf{u}|^2 + |A\mathbf{u}|^2 &\leq \frac{\kappa_2}{Pr} |A\mathbf{u}|^{3/2} |\nabla \mathbf{u}|^{3/2} + Ra |T| |A\mathbf{u}| \\ &\leq \frac{1}{2} |A\mathbf{u}|^2 + \frac{\kappa_3}{2Pr^4} |\nabla \mathbf{u}|^6 + Ra^2 |T|^2 \end{aligned}$$

which implies

$$(25) \quad \frac{1}{Pr} \frac{d}{dt} |\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^2 \leq \frac{\kappa_3}{Pr^4} |\nabla \mathbf{u}|^6 + 2\kappa_1^2 Ra^2$$

where we have used the following well-known inequality on the nonlinear term

$$(26) \quad \left| \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot A \mathbf{u} \right| \leq \kappa_2 |A \mathbf{u}|^{3/2} |\nabla \mathbf{u}|^{3/2}.$$

Recall that it is suffice to derive a bound for $|\nabla \mathbf{u}|$ uniform for large time. For this purpose we notice that $|\nabla \mathbf{u}|$ is non-increasing if the following are satisfied

$$\begin{aligned} |\nabla \mathbf{u}|^2 &\geq \frac{2\kappa_3}{Pr^4} |\nabla \mathbf{u}|^6, \\ |\nabla \mathbf{u}|^2 &\geq 4\kappa_1^2 Ra^2 \end{aligned}$$

which is equivalent to

$$\begin{aligned} |\nabla \mathbf{u}|^2 &\leq \frac{Pr^2}{\sqrt{2\kappa_3}}, \\ |\nabla \mathbf{u}|^2 &\geq 4\kappa_1^2 Ra^2. \end{aligned}$$

This implies that the ball of radius

$$(27) \quad R_0 = \frac{Pr}{(2\kappa_3)^{1/4}}$$

in H^1 is invariant provided that

$$(28) \quad \frac{Ra}{Pr} \leq \kappa_4 = \frac{1}{2(2\kappa_3)^{1/4}\kappa_1}.$$

In order to show that the ball with radius R_0 is an absorbing ball, we need to show that all suitable weak solutions enter this ball at sometime. This is an easy consequence of (24), (27), (28) and by taking T^* large enough. Indeed, for large enough T^* , we have

$$\frac{1}{T^* Pr} |\mathbf{u}_0|_{L^2}^2 \leq \frac{Pr^2}{2\sqrt{2\kappa_3}}$$

which implies, for large enough T^* ,

$$(29) \quad \frac{1}{T^* Pr} |\mathbf{u}_0|_{L^2}^2 + Ra^2 \kappa_1^2 (|T_0|) \leq \frac{Pr^2}{2\sqrt{2\kappa_3}} + Pr^2 \kappa_4^2 \kappa_1^2 = \frac{3}{4} \frac{Pr^2}{\sqrt{2\kappa_3}} = \frac{3}{4} R_0^2$$

This implies eventual regularity for the velocity field on or before T^* . The eventual regularity for the temperature field then follows (see next section).

For the eventual regularity result stated in Theorem 1, we need a refined estimate on κ_4 so that it is independent of initial data. This will be part of the proof of the existence of global attractor as we shall discuss in the sequel.

4. Existence of global attractor

The purpose of this section is to give a sketch of the existence of a global attractor for the Boussinesq system at large Prandtl number. The global attractor is regular (in fact analytic) and it attracts all suitable weak solutions.

In order to derive the existence of global attractor, we only need to establish the existence of a bounded absorbing ball in H^1 which attracts any ball in the phase space in finite time (depending on the radius of the ball only). For our particular case, it is suffice to derive an absorbing ball in H^1 for the velocity field only since the absorbing ball for the temperature follows easily. Indeed, assuming the existence of

an absorbing ball in H^1 for the velocity field, we recall the temperature equation in the perturbative variable as

$$(30) \quad \frac{\partial}{\partial t}\theta + \mathbf{u} \cdot \nabla\theta - u_3 = \Delta\theta.$$

Multiplying the equation by θ , integrating over Ω , utilizing Cauchy-Schwarz and Poincaré inequality we have

$$\frac{d}{dt}|\theta|_{L^2}^2 + |\theta|_{L^2}^2 + |\nabla\theta|_{L^2}^2 \leq 2 \int_{\Omega} u_3\theta \leq \kappa_5(|\mathbf{u}_0|, |T_0|)$$

which further implies

$$(31) \quad e^{-t} \int_0^t e^s |\nabla\theta(s)|_{L^2}^2 ds \leq \kappa_5(|\mathbf{u}_0|, |T_0|)$$

Next, we multiply the perturbative temperature equation by $-\Delta\theta$ and integrate over Ω . We then have,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\nabla\theta|_{L^2}^2 + |\Delta\theta|_{L^2}^2 &\leq |u_3|_{L^2} |\Delta\theta|_{L^2} + |\mathbf{u}|_{L^6} |\nabla\theta|_{L^3} |\Delta\theta|_{L^2} \\ &\leq \frac{1}{2} |\Delta\theta|_{L^2}^2 + |u_3|_{L^2}^2 + \kappa_6 |\nabla\mathbf{u}|_{L^2}^4 |\nabla\theta|_{L^2}^2 \end{aligned}$$

which further implies that

$$(32) \quad |\nabla\theta(t)|_{L^2}^2 \leq e^{-t} |\nabla\theta_0|_{L^2}^2 + 2 \|u_3\|_{L^\infty(0,t;L^2)}^2 + \kappa_6 \kappa_5(|\mathbf{u}_0|, |T_0|) \|\nabla\mathbf{u}\|_{L^\infty(0,t;L^2)}^2.$$

Recall that every suitable weak solution must enter a ball of radius R_0 specified in (27) after finite time (depending on the L^2 norm of the initial data) provided (28) is satisfied. Notice that the right hand side of (28) depends on the initial data which is undesirable here. The dependence on the initial data can be eliminated via the following procedure. Observe, from (20-21) that for the temperature field there exists a time T_1^* depending on the L^2 norm of the initial temperature, such that

$$(33) \quad |T(t)|_{L^2} \leq 2|\Omega| = \kappa_1^*, \forall t \geq T_1^*.$$

Thus we may consider the problem starting from T_1^* and we have uniform H^1 bound for large time for the velocity field provided that (28) is satisfied with $\kappa_1(|T_0|)$ replaced by κ_1^* (a function of the domain only). We may then find a T_2^* ($> T_1^*$) (depending on the L^2 norm of the initial data only) such that the velocity field enter the absorbing ball of radius R_0 between T_1^* and T_2^* utilizing estimates similar to (22-24). This further implies uniform H^1 bound on the temperature field at large time. These are suffice for the existence of global attractor by a simple modification of the classical theory (see for instance Temam 1997). The result here is different from well-known results for attractors for 3D flows in the sense that the flows on the global attractor are classical flows and no additional assumptions are imposed (see Temam 1997, Ball 2000, among others). Once the existence of a global attractor is established, the smoothness of solutions on the global attractor is easy to establish via standard bootstrap argument (see for instance Temam 1997). We leave the detail to the interested reader.

One may wonder if the global attractor is trivial in this large Prandtl number case since large Prandtl number is equivalent to (relatively) large viscosity. It turns out that the attractor could be extremely complicated. It contains at least three steady states: the pure conduction state which exists for all Rayleigh number, and

two Bénard cells which exist when the Rayleigh number crosses a threshold value (see for instance Rabinowitz 1968 and Yudovitch 1967, and for a contemporary account, see Ma and Wang 2004). Experiments at large Prandtl number fluids indicate that the long time behavior of the flows are still very complicated (thermal plumes may pop up at seemingly random location at random time, see Tritton 1988 among others). The exact structure of the global attractor is unknown and remains a challenge at large Rayleigh number. It is possible to show that the global attractor of the Boussinesq system converge to the global attractor of the infinite Prandtl number model (formally setting the Prandtl number to infinite) in an appropriate sense (upper-semi continuous) (Wang 2003a) after refining estimates here. However, even the global attractor of the simplified infinite Prandtl number model could be extremely complicated (see Wang 2003b).

Now we have established the existence of a global attractor which attracts all suitable solutions. At the first glance, it seems that the radius of the absorbing ball depends on the Prandtl number Pr (see (27)), and it approaches infinity as the Prandtl number approaches infinity. This will prevent us from taking limit as the Prandtl number approaches infinity for the long time behavior. However this unfavorable dependence on the Prandtl number can be eliminated by noting that within the absorbing ball, inequality (25) implies

$$(34) \quad \frac{1}{Pr} \frac{d}{dt} |\nabla \mathbf{u}|^2 + \frac{1}{2} |\nabla \mathbf{u}|^2 \leq 2(\kappa_1^*)^2 Ra^2$$

at large time (so that κ_1 may be replaced by κ_1^*). This leads to an absorbing ball in H^1 with radius R_1 given by

$$(35) \quad R_1 = 2\kappa_1^* Ra$$

This and other uniform (in Prandtl number) estimates plays a central role in studying asymptotic statistical behavior of the Rayleigh-Bénard system at large Prandtl number (Wang 2003a).

5. Existence of suitable weak solution

We now answer the question if suitable weak solutions as defined earlier exist.

Notice that the extra inequalities (20-21) on the temperature field are apparently satisfied for smooth solutions. Indeed, such maximum principle type inequalities have been established for two dimensional convective flows where regularity of solutions are known (see Foias, Manley and Temam 1987). Here we take the standard approach of regularizing the problem and taking appropriate limit. Thus we consider the problem

$$(36) \quad \frac{1}{Pr} \left(\frac{\partial \mathbf{u}^\varepsilon}{\partial t} + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{u}^\varepsilon \right) + \nabla p^\varepsilon = \Delta \mathbf{u}^\varepsilon - \varepsilon \Delta^2 \mathbf{u}^\varepsilon + Ra \mathbf{k} T^\varepsilon, \quad \nabla \cdot \mathbf{u}^\varepsilon = 0,$$

$$(37) \quad \frac{\partial T^\varepsilon}{\partial t} + \mathbf{u}^\varepsilon \cdot \nabla T^\varepsilon = \Delta T^\varepsilon,$$

$$(38) \quad \mathbf{u}^\varepsilon|_{z=0,1} = \Delta \mathbf{u}^\varepsilon|_{z=0,1} = 0,$$

$$(39) \quad T^\varepsilon|_{z=0} = 1, \quad T^\varepsilon|_{z=1} = 0.$$

It is easy to check, following classical energy estimates (see for instance Lions 1969 among others) that solutions to this regularized problem are regular and satisfy all the inequalities (19-21) in the definition of suitable weak solutions independent of

the artificial hyper-viscosity ε . We then pass to the limit and obtain suitable weak solutions.

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