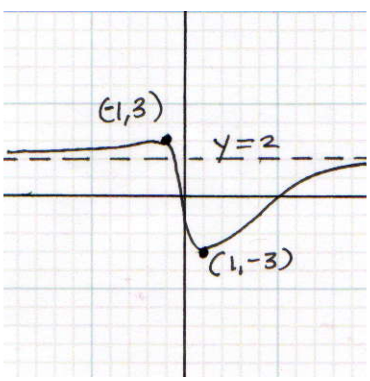
Limits at Infinity

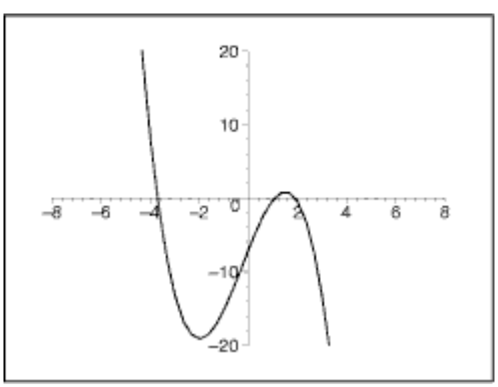
Definition 4.4.1. *We say the limit as x approaches infinity is L, written $\lim_{x\to\infty} f(x) = L$, if for some x large enough the graph of $y = f(x)$ moves closer and closer to the line $y = L$ as one moves to the right. Moreover, in this case the graph $y = f(x)$ has a horizontal asymptote $y = L$. (L must be a real number!)*

Definition 4.4.2. *We say the limit as x approaches negative infinity is L, written $\lim_{x\to-\infty} f(x) = L$, if for some x far enough to the left the graph of $y = f(x)$ moves closer and closer to the line $y = L$ as one moves further to the left. Moreover, in this case the graph $y = f(x)$ has a horizontal asymptote $y = L$. (L must be a real number!)*

Example 4.4.1. *Find* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$ *for the following function.*



Example 4.4.2. *Find* $\lim_{x\to\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$ *for the following function.*



Steps for Determining Limits at Infinity Given Equations

(1) If $f(x)$ is a polynomial, then $\lim_{x\to\pm\infty} f(x) = \pm\infty$. The sign is determined by the leading term.

(2) If $f(x)$ is a rational function, then
$\lim_{x\to\pm\infty} f(x) = \pm\infty$ if degree of numerator is higher than degree of denominator (Use step 1 to determine plus or minus)

$\lim_{x\to\pm\infty} f(x) = 0$ if degree of denominator is higher than degree of numerator

$\lim_{x\to\pm\infty} f(x) = a/b$ if degree of numerator is equal to degree of denominator and a is the leading coefficient of numerator and b is leading coefficient of denominator.

Example 4.4.3. *Find the limits* $\lim\limits_{x\to\infty} 4x^5$ *and* $\lim\limits_{x\to-\infty} 4x^5$

Example 4.4.4. *Find the limits* $\lim\limits_{x\to\infty} -4x^5$ *and* $\lim\limits_{x\to-\infty} -4x^5$

Example 4.4.5. *Find the limits* $\lim\limits_{x\to\infty} -4x^8$ *and* $\lim\limits_{x\to-\infty} -4x^8$

Example 4.4.6. *Find the limits* $\lim\limits_{x\to\infty} (-2x^3 + 4x^2 - x + 5)$ *and* $\lim\limits_{x\to-\infty} (-2x^3 + 4x^2 - x + 5)$

Example 4.4.7. *Find the limits* $\lim\limits_{x\to\infty} (-2x^3 + 4x^5 - x + 5)$ *and* $\lim\limits_{x\to-\infty} (-2x^3 + 4x^5 - x + 5)$

Example 4.4.8. *Find the limits* $\lim_{x\to\infty} \frac{1}{x}$ *and* $\lim_{x\to-\infty} \frac{1}{x}$

Example 4.4.9. *Find the limit* $\lim_{x\to-\infty} \frac{-2x^2+2}{5x^3+x^2-1}$

Example 4.4.10. *Find the horizontal asymptote(s) of* $f(x) = \frac{-2x^2+2}{5x^3+x^2-1}$

Example 4.4.11. *Find the limit* $\lim_{x\to\infty} \frac{-2x^3+x^2-1}{3x^2+2}$

Example 4.4.12. *Find the horizontal asymptote(s) of* $f(x) = \frac{-2x^3+x^2-1}{3x^2+2}$

Example 4.4.13. *Find the limit* $\lim_{x\to\infty} \frac{3x^5+2x^3-1}{6x^5+7}$

Example 4.4.14. *Find the horizontal asymptote(s) of* $f(x) = \frac{3x^5+2x^3-1}{6x^5+7}$

Example 4.4.15. *Select ALL the correct choices*

(1) $\lim_{x\to\infty} 4x^4 - 3x^2 + 7 = \infty$

(2) $\lim_{x\to\infty} 4x^4 - 3x^2 + 7 = -\infty$

(3) $\lim_{x\to-\infty} 4x^4 - 3x^2 + 7 = \infty$

(4) $\lim_{x\to-\infty} 4x^4 - 3x^2 + 7 = -\infty$

(5) $\lim_{x\to\infty} \frac{3x^2 - 4x + 2}{x - 6} = -\infty$

(6) $\lim_{x\to-\infty} \frac{4x^3 - 6x + 1}{5 - x^2} = \infty$

(7) $\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{4 - x^2} = 3$

(8) $\lim_{x\to\infty} \frac{x^4 - 2x}{5 - x^4} = -1$

(9) $\lim_{x\to\infty} \frac{4 - x^3}{6 - x^4} = 1$

Infinite Limits and Vertical Asymptotes (review from section 4.1)

Definition 4.4.3. *If* $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a^-} f(x) = \pm\infty$, *or* $\lim_{x\to a^+} f(x) = \pm\infty$, *then the vertical line* $x = a$ *is a* **vertical asymptote** *of the curve* $y = f(x)$.

Example 4.4.16. *Find the vertical asymptote(s) of* $f(x) = \frac{x^2 - x - 6}{x^2 - 9}$ *(be careful!)*

Example 4.4.17. *Select ALL the correct choices*

(1) $f(x) = \dfrac{x^2-1}{x+3}$ *has a vertical asymptote at* $x = -3$

(2) $f(x) = \dfrac{x^2-1}{x+3}$ *has no vertical asymptote*

(3) $f(x) = \dfrac{2x}{x^2-3x}$ *has a vertical asymptote at* $x = 0$ *and* $x = 3$

(4) $f(x) = \dfrac{2x}{x^2+9}$ *has a vertical asymptote at* $x = -3$ *and* $x = 3$

(5) $f(x) = \dfrac{x^2-1}{x+3}$ *has a horizontal asymptote at* $y = -3$

(6) $f(x) = \dfrac{x^2-1}{x+3}$ *has no horizontal asymptote*

(7) $f(x) = \dfrac{x+3}{x^2+4x+3}$ *has a horizontal asymptote at* $y = 0$

(8) $f(x) = \dfrac{x+3}{x^2+4x+3}$ *has a horizontal asymptote at* $y = -1$

(9) $f(x) = \dfrac{4x-3}{2-x}$ *has a horizontal asymptote at* $y = -4$

Oblique Asymptotes

If $R(x) = \frac{p(x)}{q(x)}$ is a rational function then R may have a **oblique asymptote** (also called a **slant asymptote**). To determine if one exists and find one we consider the degrees of p and q. Note that a horizontal asymptote describes the *end behavior* of R.

- If the degree of p is equal to

To find the oblique asymptote of a rational function of this form, you have to use long division of polynomials.

Example 4.4.18. *Find all the asymptote(s) of* $f(x) = \dfrac{4x^2 - 2x + 4}{2x + 1}$

Graphing Strategy

(1) Find from $y = f(x)$:

(a) Domain: where is f defined? (Do NOT simplify before finding domain)

(b) x-intercepts: set $y = 0$ and solve for x

(c) y-intercepts: set $x = 0$ and solve for y

(d) Asympotes

(i) Vertical: find a so that $\lim_{x \to a} f(x) = \pm\infty$.

(ii) Horizontal: find L so that $\lim_{x\to\pm\infty} f(x) = L$.

(iii) Slant or Oblique: If $f(x) = r(x)/s(x)$ is a rational expression where the degree of the numerator is one more than the degree of the denominator, then there is a slant asymptote whose equation is the quotient of $r(x)/s(x)$.

(2) Find from $y = f'(x)$:

(a) Critical Numbers: where is $f'(x)$ equal to 0 or undefined in the domain of $f(x)$.

(b) Horizontal and Vertical Tangents of $f(x)$

(c) Intervals of increase and Interval of decrease of $f(x)$: use the sign of $f'(x)$

(d) Local Extrema of $f(x)$

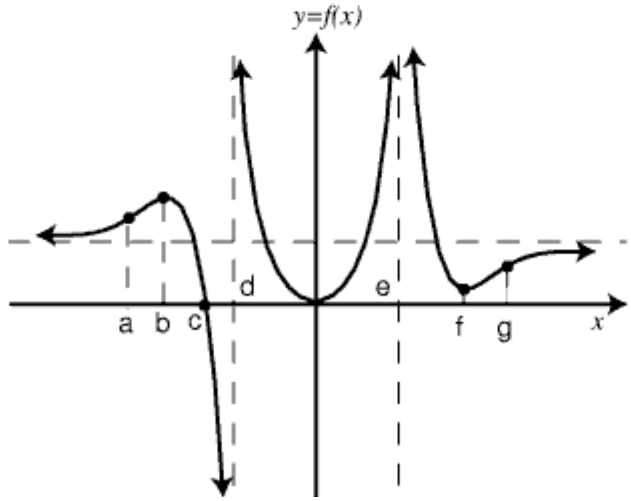
(3) Find from $y = f''(x)$:

(a) Intervals of Concave Up and Concave down of $f(x)$: use the sign of $f''(x)$

(b) Inflection Points of $f(x)$: where does $f''(x)$ change signs?

Examples

Example 4.4.19. *Assuming f', f'' exist, select ALL the correct choices for the graph.*



(1) $f''(x) > 0$ *on* $(-\infty, b) \cup (d, e) \cup (e, f)$

(2) $f''(x) < 0$ *on* $(b, d) \cup (f, \infty)$

(3) $f'(x) < 0$ *on* (c, d) *only*

(4) $f'(x) > 0$ *on* $(-\infty, c) \cup (d, e) \cup (e, \infty)$

(5) *the graph has inflection points at* $x = b$, $x = 0$, *and* $x = f$

(6) *the graph of f is concave downward on* $(a, d) \cup (g, \infty)$

(7) $f(x)$ *has extremum at* $x = b$, $x = 0$, *and* $x = f$

(8) $f'(x)$ *has extremum at* $x = b$, $x = 0$, *and* $x = f$

(9) $f'(x)$ *is increasing on* $(-\infty, c) \cup (d, e) \cup (e, \infty)$

(10) $f'(x)$ *is decreasing on* $(-\infty, d)$

Example 4.4.20. *Sketch the graph* $y = \dfrac{x^3 - 1}{x^3 + 1}$, $y' = \dfrac{6x^2}{(x+1)^2(x^2 - x + 1)^2}$,

$$y'' = -\frac{12x(2x^3 - 1)}{(x+1)^3(x^2 - x + 1)^3}$$

Example 4.4.21. *Sketch the graph* $y = x^{5/3} - 5x^{2/3}$, $y' = \dfrac{5(x-2)}{3x^{1/3}}$, $y'' = \dfrac{10(x+1)}{9x^{4/3}}$

Example 4.4.22. *Sketch the graph of* $y = \dfrac{x^3 - x}{x^2 + 3x + 2}$. *Include domain, asymptotes and intercepts.*

Homework: 4.4 p. 287 # 1-14, 15, 19, 23, 27, 33, 39, 43, 83 work e-grade practice at least 2 times.