4.5. Optimization

DEFINITIONS

$y = f(x)$ is a function with domain D.

(1) f has an ______________________ or ______________________

at $x = c$ if $f(c) \geq f(x)$ for all x in D. $f(c)$ is the ______________________.

(2) f has an ______________________ or ______________________

at $x = c$ if $f(c) \leq f(x)$ for all x in D. $f(c)$ is the ______________________.

(3) The minimum and maximum values are called the ______________________ of f.

(4) f has an ______________________ or ______________________

at $x = c$ if $f(c) \geq f(x)$ for x "close enough" to c. $f(c)$ is the ______________________ .

(5) f has an ______________________ or ______________________

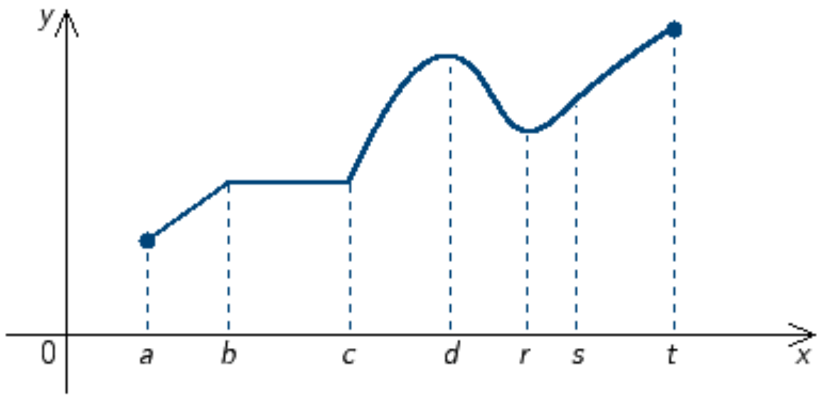
at $x = c$ if $f(c) \leq f(x)$ for x "close enough" to c. $f(c)$ is the ______________________.

(6) "close enough" to c means there is an open interval around c where the statement is true. This open interval can be very small.

(7) The local minimum and local maximum values are called the ______________________ of f.

4.5. Examples

Example 4.5.1. *Find all absolute and local extrema and where they occur.*



Example 4.5.2. *Select ALL the correct choices* $f(x) = 2 - 4x - \dfrac{4}{x}$ *over the interval* $(-\infty, 0)$

(1) $f(x)$ *has no maximum*
(2) $f(x)$ *has no minimum*
(3) $f(x)$ *has an absolute maximum at* $x = -1$
(4) $f(x)$ *has an absolute minimum at* $x = -1$
(5) $f(x)$ *has an absolute maximum of* 2
(6) $f(x)$ *has an absolute minimum of* 2

Useful Theorems

The Extreme Value Theorem: If f is continuous on the closed interval $[a, b]$, then f will attain a minimum and a maximum in the interval.

In other words, if you consider the interval $[a, b]$ as the domain of f, there will be at least one number c in $[a, b]$ where $f(c)$ is the maximum, and at least one number d in $[a, b]$ where $f(d)$ is the minimum.

Closed Interval Method

To find the *absolute* minimum and maximum values of a *continuous* function f on a *closed interval* $[a, b]$:

Step 1. Find the critical numbers of f in (a, b).

Step 2. Find the function value at all critical value(s) found in step 1.

Step 3. Find $f(a)$ and $f(b)$.

Step 4. The largest value from steps 2 and 3 is the maximum value and the smallest value from steps 2 and 3 is the minimum value.

Examples

Example 4.5.3. *Find all critical values and absolute extrema on the given interval.*

$f(x) = 6x - x^2$, $[-1, 4]$

(1) min value is -7*, max value is* 9
(2) min value is -7*, max value is* 40
(3) min value is -5*, max value is* 8
(4) min value is -5*, max value is* 40

Example 4.5.4. *Find all critical values and absolute extrema on the given interval.*

$$f(x) = \frac{x^2 - 4}{x^2 + 4},\ [-4, 4]$$

Example 4.5.5. *Find all the absolute extrema of* $f(x) = 1 + 9x + \frac{4}{x}$ *over the interval* $[-2, -1]$.

4.7 Optimization - main steps

Step 1. Read problem and express all information from the problem mathematically. Use variables to represent any quantity that changes. Numbers may be used for quantities that remain constant.

Step 2. Find a function for the quantity to be optimized in terms of one variable.

Step 3. Find the absolute extreme required using the techniques discussed in section 4.1.

Step 4. re-read the problem and answer the question.

Optimization Examples

Example 4.5.6. *If the price-demand and cost functions for a product are* $p = 500 - x$ *and* $C(x) = \frac{1}{2}x^2 + 200x + 55$*, respectively, how many units,* x*, will maximize the profit?*

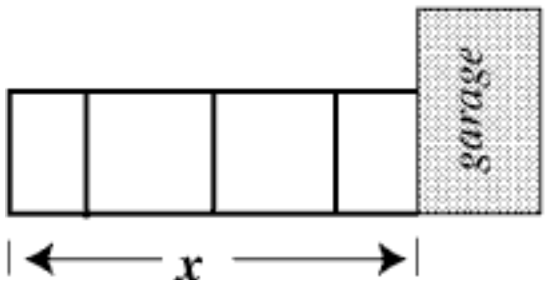
Example 4.5.7. *If a farmer plants 50 trees per acre, each tree will yield 150 bushels of peaches. For each additional tree he plants per acre, the yield of each tree will decrease by 3 bushels. If* x *is the number of ADDITIONAL trees planted per acre, find the total yield per acre in terms of* x*. Find the maximum yield.*

Example 4.5.8. *An airline sells 200 tickets per day at* \$100 *per ticket. Each* \$5 *price reduction will result in 20 more tickets sold per day. If x is the number of* \$5 *price reductions,*

(1) find the total revenue, R, in terms of x.

(2) find the value for x that maximizes revenue.

Example 4.5.9. *A homeowner has 6400 feet of fencing material to use in enclosing three adjacent rectangular pens next to the garage. The side along the garage will not need fencing. (see figure) Find the total area of the enclosure $A(x)$, in terms of x.*



Example 4.5.10. *A homeowner has 6400 feet of fencing material to use in enclosing three adjacent rectangular pens next to the garage. The side along the garage will not need fencing. (see figure in example 4.5.9) Find the maximum total area of the enclosure $A(x)$*

Example 4.5.11. *A square piece of cardboard 5 inches on each side is to be made into a box by cutting squares with length x from each corner and folding up the sides.*

(1) *Find the volume of the box in terms of x.*
(2) *Find the values of x that maximizes the volume.*
(3) *find the maximum volume.*

Homework: 4.5 p. 304 # 3, 9, 13, 17, 19, 23, 37, 43, 45, 47, 49 work e-grade practice at least 2 times.