2.4. The Derivative

Definition 2.4.1. *Recall from 2.1: Given a function* $y = f(x)$, *a* **difference quotient** *is an expression of the form*

$$\frac{f(a+h) - f(a)}{(a+h) - h}$$

Definition 2.4.2. *The* **average rate of change from** $x = a$ **to** $x = a + h$ *is*

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \qquad h \neq 0$$

Definition 2.4.3. *The* **average rate of change from** $x = a$ **to** $x = b$ *is*

$$\frac{f(b) - f(a)}{b - a}$$

Remark 2.4.1. *All of the following concepts are found using the average rate of change:*

(1) the slope of a secant line,
(2) average velocity of a particle,
(3) average acceleration of a particle using velocity

Example 2.4.1. *Find the indicated quantities for* $f(x) = 16x^2$.

A) The average rate of change from $x = 2$ *to* $x = 6$.

B) The average rate of change from $x = 2$ *to* $x = 2 + h$.

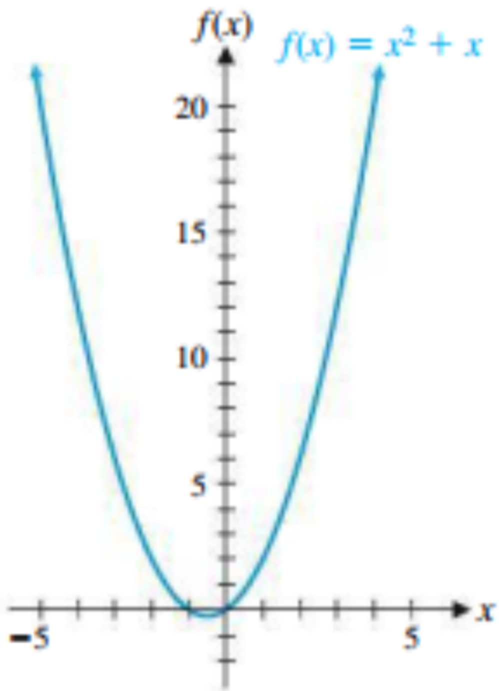
Example 2.4.2. *If* $f(x) = 3 - 2x^2$, *find* $\dfrac{f(2) - f(-5)}{2 - (-5)}$.

Example 2.4.3. *Given* $f(x) = 2 - 2x - x^2$, *find* $\dfrac{f(-1+h) - f(-1)}{h}$, $h \neq 0$.

Example 2.4.4. *Given* $f(x) = 2 - 2x - x^2$, *find* $\lim_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h}$.

Definition of the Derivative

Example 2.4.5. *Refer to the following graph of $f(x) = x^2 + x$ to answer the problems below.*



(1) *Find the slope of the secant line adjoining the points at $x = 1$ and $x = 3$.*

(2) *What is the slope of $f(x)$ at $x = 1$?*

Definition 2.4.4. *The line ________________________ to a curve at a point $x = a$ is the line the "best approximates" the curve at $x = at$. The slope of this line at $x = a$ is the slope of the curve at $x = a$.This slope is also called the* **instantaneous rate of change**. *The formal definition is given next.*

Definition 2.4.5. *Let $y = f(x)$ be a function. The* **derivative of f with respect to x** *is*

$$f'(x) = y' = \frac{d}{dx}f(x) = \frac{dy}{dx} = Df(x) = D_x f(x) =$$

Remark 2.4.2. *All of the following concepts are found using the derivative:*

(1) *the slope of a tangent line,*
(2) *(instantaneous) velocity of a particle at a particular time using the position,*
(3) *(instantaneous) acceleration of a particle at a particular time using velocity,*
(4) *instantaneous rate of change of a quantity*
(5) *marginal cost using a cost function*
(6) *marginal revenue using a revenue function*

Example 2.4.6. *Find the slope of* $f(x)$ *from Example 2.4.5 at* $x = 1$*:*

Example 2.4.7. *Given* $f(x) = 2 - 2x - x^2$*, find* $f'(x)$.

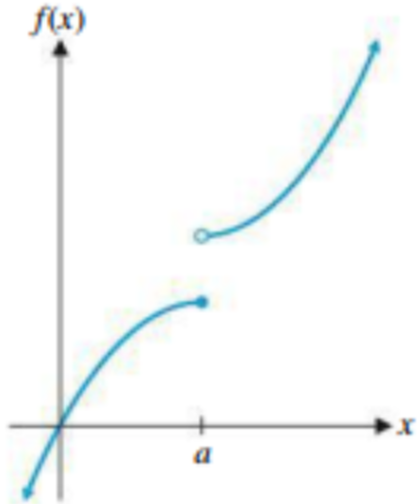
Example 2.4.8. *Given* $f(x) = 1/x$*, find* $f'(x)$*. Then find* $f'(1)$ *and* $f'(2)$*.*

Example 2.4.9. *The total sales of a company (in millions of dollars)* t *months from now are given by* $S(t) = \sqrt{t+6}$*. Find* $S(10)$ *and* $S'(10)$*, and interpret. Use these results to estimate the total sales after 13 months and 14 months.*

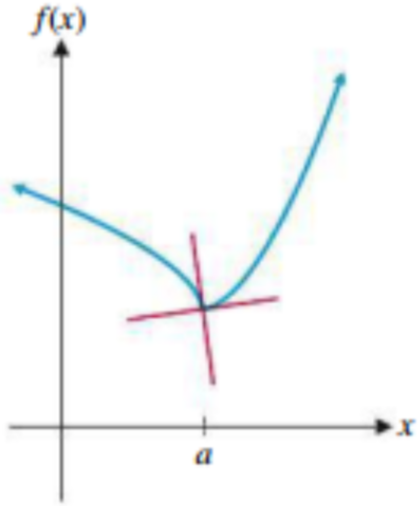
Determining where $f(x)$ is Nondifferentiable

Definition 2.4.6. *We say that $f(x)$ is* **nondifferentiable** *at $x = a$ if the derivative of $f(x)$ does not exist at $x = a$. This happens when*

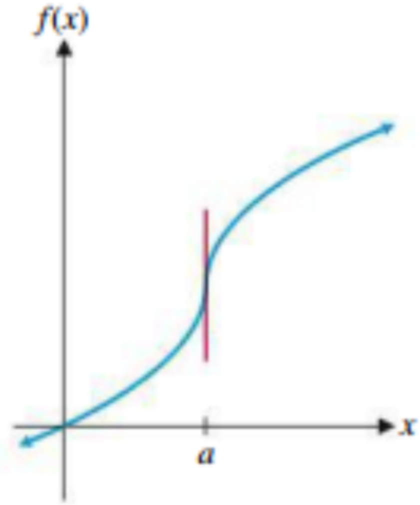
(1) the graph of $f(x)$ has a hole, break, or vertical asymptote at $x = a$
(2) the graph of $f(x)$ has a sharp corner at $x = a$
(3) the graph of $f(x)$ has a vertical tangent at $x = a$



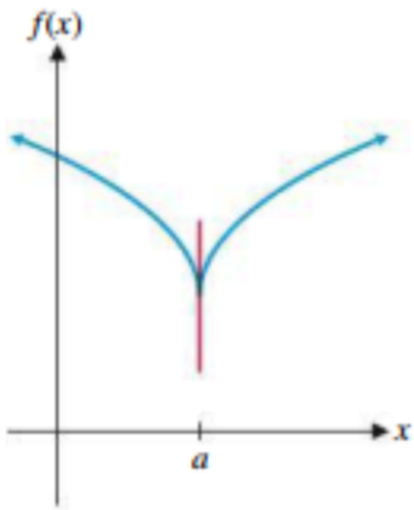
(A) Not continuous at $x = a$



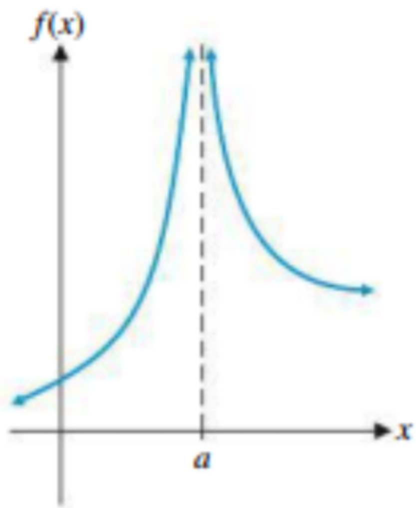
(B) Graph has sharp corner at $x = a$



(C) Vertical tangent at $x = a$



(D) Vertical tangent at $x = a$



(E) Vertical asymptote at $x = a$

Example 2.4.10. *Determine where $f(x)$ is nondifferentiable given its graph below.*